The Faculty of Mathematics at the University of Waterloo
in association with
The Centre for Education in Mathematics and Computing
presents

The Eighth Annual Small c Competition

for First and Second Year Students

Friday 33 September 2008

Time: 1 hour
Calculators are permitted.
Instructions:

1. Do not open the Contest booklet until you are told to do so.

10. You may use slide rules, abaci, rulers, compasses and paper for rough work. You may also use log tables; log cabins are not permitted. This year, Tom-toms and Coleman stoves are again permitted. Protractors are also permitted, though contractors are not; if you can find a sub-contractor during this construction boom, by all means, use him/her.

11. Any contestant carrying an Elongated Pentagonal Orthocupolarotunda must register it with a proctor.

100. On your response form, print your name, plan, and ID number.

101. This is a multiple choice test. Each question is followed by five possible answers marked **A**, **B**, **C**, **D**, and **E**. Only one of these is correct. When you have decided on your choice, fill in the appropriate bubble on the response form.

110. In the past, your response form was read only by a *dumb human*, who had undergone rigorous training in order to be able to recognize the letters **A** through **E**. Due to labour unrest, this year, the dumb humans have been replaced by even dumber machines.

111. Scoring: Each correct answer is worth 5 in Part A, 6 in Part B, and 8 in Part C.
There is *no penalty* for an incorrect answer.
Each unanswered question is worth 2, to a maximum of 20.

1000. Diagrams are *not* drawn to scale. They are intended as aids only.

1001. Als u dit kunt lezen, spreekt u het Nederlands.

1010. When your supervisor instructs you to begin, you will have *sixty* minutes of working time.

1011. Please do not sing aloud while doing #16.

Part A

1. The value of $1 + 3 + 5 + 7 + 9$ is

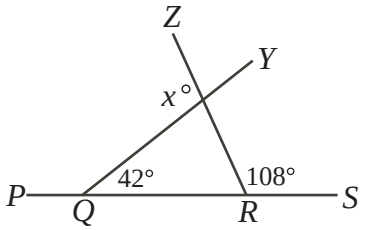
   (A) 2^2 **(B)** 6^2 **(C)** 4^2 **(D)** 7^2 **(E)** 5^2

2. On his summer holiday, Faculty of Math Dean Tom Coleman bought 10 pink ties for \$12 each and 8 construction pylons for \$9 each. If he started with \$200, how much money did he have left after his purchases?

   (A) \$12 **(B)** \$8 **(C)** \$14 **(D)** \$18 **(E)** \$10

3. In the diagram, PS, QY and RZ are straight line segments. The value of x is

   (A) 114 **(B)** 108 **(C)** 138
   (D) 120 **(E)** 126



4. The expression $1 + 2(14) + 14^2$ is equal to

   (A) $(2^2)(7^2)$ **(B)** 28^2 **(C)** 15^2 **(D)** 24^2 **(E)** 13^2

5. A sequence $\{x_n\}$ is defined by $x_1 = 4$, $x_2 = 5$ and $x_{k+1} = 2x_k + x_{k-1}$ for $k \geq 2$. The term x_4 equals

   (A) 80 **(B)** 23 **(C)** 31 **(D)** 24 **(E)** 33

6. WHICH CAPITAL LETTER DOES NOT OCCUR MORE THAN TWICE IN THIS QUESTION, INCLUDING IN THE FIVE OPTIONS?

   (A) A **(B)** B **(C)** C **(D)** D **(E)** E

7. There are 6 three-digit positive integers that can be formed using each of the digits 1, 6 and 9 exactly once. How many of these 6 integers are perfect squares?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

8. If x is a real number, then $\lfloor x \rfloor$ is the greatest positive integer less than or equal to x. For example, $\lfloor 4.5 \rfloor = 4$ and $\lfloor -2.1 \rfloor = -3$. The value of $\left\lfloor \frac{1}{7} \left\lfloor \frac{1777}{17} \right\rfloor \right\rfloor$ is

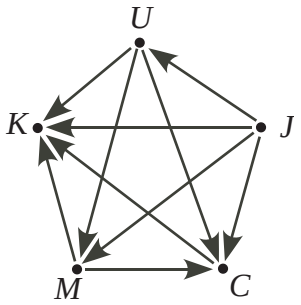
   (A) 15 **(B)** 14 **(C)** 16 **(D)** 19 **(E)** 17

9. The average of three numbers is 13. Two numbers are added to this list so that the average of all five numbers is 17. What is the average of the two new numbers?

   (A) 21 **(B)** 25 **(C)** 23 **(D)** 30 **(E)** 15

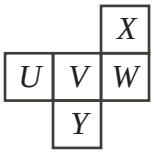
10. The network on the right illustrates the relative ages of five children: Ugueth, Kimihiro, Jearlyn, Mohammed, and Chen. The arrow from U to K means that U is older than K. Which is the correct order of ages, youngest first?

    (A) $JUCMK$ **(B)** $UMCJK$ **(C)** $KCMUJ$
    (D) $KCUMJ$ **(E)** $JUMCK$



Part B

11. The net in the diagram is folded to make a box with an open top. When the box is placed on a table with its open top pointing up, the letter on the bottom face is

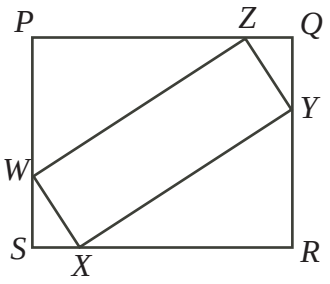


(A) U **(B)** V **(C)** W
(D) X **(E)** Y

12. If a, b, c, d are non-negative integers with $a(1!) + b(2!) + c(3!) + d(4!) = 85$ and $a \leq 1$ and $b \leq 2$ and $c \leq 3$ and $d \leq 4$, then $a + b + c + d$ is equal to

(A) 10 **(B)** 6 **(C)** 9 **(D)** 4 **(E)** 7

13. In the diagram, $PQRS$ is a rectangle. Also, the vertices of rectangle $WXYZ$ are on the edges of rectangle $PQRS$, as shown. If $WX = 5$, $XY = 25$ and $SX = 3$, then $PQ - PS$ equals



(A) 9 **(B)** 6 **(C)** 3
(D) 4 **(E)** 5

14. A UW mathematics professor (from an unnamed department) owns five single socks. Each sock is of course a different colour. Each day, he wears two socks (including, obviously, the days on which he wears sandals). He wears each sock an equal number of days over the course of a 365 day year. This number of days is

(A) 365 **(B)** 73 **(C)** 183 **(D)** 146 **(E)** 150

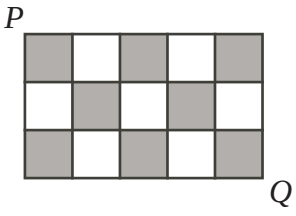
15. A real number p satisfies $0 < p < 2$. Which of the following is always true?

(A) $p^2 > 4$ **(B)** $p < \dfrac{2}{p}$ **(C)** $p^2 - 2p > p$ **(D)** $p^3 < 4p$ **(E)** $p^2 - 2p > 3$

16. All together now: "The Grand Old Duke of York, he had ten thousand men, he marched them up to the top of the hill..." By 2:00 p.m., they were one-third of the way up. By 4:00 p.m., they were three-quarters of the way up. They march at a constant speed and do not take breaks. When did they start marching?

(A) 1:12 p.m. **(B)** 12:00 noon **(C)** 12:24 p.m. **(D)** 1:36 p.m. **(E)** 1:48 p.m.

17. In the diagram, each of the 15 small squares in the grid has side length 10. A grey mouse walks from P to Q along the edges of the squares. He walks so that he always is walking with a shaded square on his left. What is the length of his shortest possible path from P to Q?



(A) 100 **(B)** 120 **(C)** 180
(D) 160 **(E)** 80

18. The number of positive integers n with $1 \leq n \leq 2008$ and $\gcd(n, 6) = 2$ is

(A) 334 **(B)** 335 **(C)** 669 **(D)** 670 **(E)** 1340

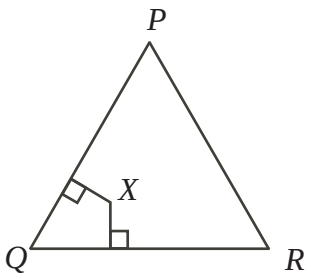
19. Consider the three statements:

P: At least one of these statements is false.
Q: Exactly two of these statements are true.
R: At least one of these statements is true.

Which of the following must be true?

(A) P is true; Q and R are false. **(B)** P, Q and R are false.
(C) P is true and exactly one of Q and R is true. **(D)** Q is true; P and R are false.
(E) There is a contradiction no matter which statements are true or false.

20. In the diagram, $\triangle PQR$ is equilateral with $PQ = 4$. Point X is inside the triangle so that the distance from X to PQ is 1 and the distance from X to QR is 1. What is the distance from X to PR?



(A) $\sqrt{3}$ **(B)** $2\sqrt{3} - 2$ **(C)** $2 - \sqrt{3}$
(D) $3 - \sqrt{3}$ **(E)** $2\sqrt{3} - 1$

Part C

21. Five valkyries, named Gerhilde, Ortlinde, Helmwige, Grimgerde, and Roßweiße, are registered in UW Computer Science. Each places her pink tie calculator in a black box. The 5 calculators are drawn out in random order and given back to the 5 valkyries. What is the probability that exactly 3 of them receive their original calculator?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{3}{20}$ **(E)** $\frac{1}{10}$

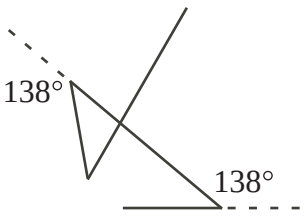
22. What fraction of the area of the circle $x^2 + (y-1)^2 = 2$ lies below the x-axis?

(A) $\dfrac{\pi - 1}{4\pi}$ **(B)** $\dfrac{\pi - 2}{4\pi}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{3\pi + 2}{4\pi}$ **(E)** $\dfrac{1}{2}$

23. The number of quadruples (a, b, c, d) of positive integers with $ab + bc + cd + da = 55$ is

(A) 80 **(B)** 88 **(C)** 108 **(D)** 72 **(E)** 60

24. Aiden is drawing a pattern on a large piece of paper. He starts by drawing a segment of length 1 cm, then turns 138° to the left and draws a segment of length 2 cm, and again turns 138° to the left. In this manner, he draws segments of length 1 cm, 2 cm, 1 cm, 2 cm, and so on with each segment drawn following a 138° turn. The first four steps are shown in the diagram. Aiden discovers that the pattern eventually starts to retrace itself exactly. What is the total length, in centimetres, of the pattern before it starts to retrace?



(A) 135 **(B)** 75 **(C)** 60
(D) 120 **(E)** 90

25. A quadruple (a, b, c, d) of positive integers such that

- $a \leq b \leq c \leq d$,
- a, b, c, d are consecutive terms in an arithmetic sequence,
- $a^3 + b^3 + c^3 = d^3$, and
- at least one of a, b, c, d between 1867 and 2008 inclusive.

is called a *quixotic quadruple*. How many quixotic quadruples are there?

(A) 136 **(B)** 140 **(C)** 134 **(D)** 142 **(E)** 138