MATH 146 Linear Algebra 1, Lecture Notes by Stephen New

Chapter 2. Solving Systems of Linear Equations

Systems of Linear Equations

2.1 Definition: Let R be a ring and let $n \in \mathbf{Z}^+$. A **linear equation** in R^n (or a linear equation in the variables $x_1, x_2, \cdots, x_n$ over R) is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

with $a = (a_1, a_2, \cdots, a_n)^T \in R^n$ and $b \in R$. The numbers a_i are called the **coefficients** of the equation. A **solution** to the above equation is a point $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ for which the equation holds. The **solution set** of the equation is the set of all solutions. Note that $x = 0$ is a solution if and only if $b = 0$, and in this case we say that the equation is **homogeneous**.

2.2 Example: Let F be a field, let $a = (a_1, a_2, \cdots, a_n)^T \in F^n$, let $b \in F$, and let S be the solution set of the linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Show that either $S = \emptyset$ (the empty set) or S is an affine space in F^n.

Solution: If $a = 0$ and $b \neq 0$ then the equation has no solution so we have $S = \emptyset$. If $a = 0$ and $b = 0$ then every $x \in F^n$ is a solution, so $S = F^n$. If $a \neq 0$ then we can choose an index k such that $x_k \neq 0$ and then we have

$$\begin{aligned}
\sum_{i=1}^{n} a_ix_i = b &\iff x_k = \tfrac{b}{a_k} - \sum_{i \neq k} \tfrac{a_i}{a_k} x_i \\
&\iff x = \left(x_1, x_2, \cdots, x_{k-1}, \tfrac{b}{a_k} - \sum_{i \neq k} \tfrac{a_i}{a_k}\, x_i, x_{k+1} \cdots, x_n\right)^T \\
&\iff x = p + \sum_{i \neq k} x_iu_i
\end{aligned}$$

where $p = \frac{b}{a_k}\, e_k$ and $u_i = e_i - \frac{a_i}{a_k}\, e_k$.

2.3 Definition: Let R be a ring and let $n, m \in \mathbf{Z}^+$. A **system of m linear equations** in R^n (or in n variables over R) is a set of m equations of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where each $a_{ij} \in R$ and each $b_j \in R$. The numbers a_{ij} are called the **coefficients** of the system. A **solution** to the above system is a point $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ for which all of the m equations hold. The **solution set** of the system is the set of all solutions. Note that $x = 0$ is a solution if and only if $b = 0$, and in this case we call the system of linear equations **homogeneous**.

2.4 Remark: We shall describe an algorithmic method for solving a given system of liner equations over a field F. We shall see that if the the solution set is not empty then it is an affine space in F^n. The algorithm involves performing the following operations on the equations in the system. These operations do not change the solution set.

(1) $E_i \leftrightarrow E_j$: interchange the i^{th} and j^{th} equations,
(2) $E_i \mapsto t\,E_j$: multiply (both sides of) the i^{th} equation by t where $0 \neq t \in F$, and
(3) $E_i \mapsto E_i + t\,E_j$: add t times (each side of) the j^{th} equation to (the same side of) the i^{th} equation E_i, where $t \in F$.

For now, we illustrate the algorithm in a particular example.

2.5 Example: Consider the system of linear equations over the field $\mathbf{Q}$.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 7 \\ x_1 + 0x_2 + 2x_3 &= 5 \\ 3x_1 - 4x_2 + 2x_3 &= 3 \end{aligned}$$

Show that the solution set is an affine space in $\mathbf{Q}^3$.

Solution: Performing operations of the above three types, we have

$$\Big(E_1 \leftrightarrow E_2 \Big) \begin{array}{r} x_1 + 0x_2 + 2x_3 = 5 \\ 2x_1 - x_2 + 3x_3 = 7 \\ 3x_1 - 4x_2 + 2x_3 = 3 \end{array} \begin{pmatrix} E_2 \mapsto E_2 - 2E_1 \\ E_3 \mapsto E_3 - 3E_1 \end{pmatrix} \begin{array}{r} x_1 + 0x_2 + 2x_3 = 5 \\ 0x_1 - x_2 - x_3 = -3 \\ 0x_1 - 4x_2 - 4x_3 = -12 \end{array}$$

$$\Big(E_2 \mapsto -E_2 \Big) \begin{array}{r} x_1 + 0x_2 + 2x_3 = 5 \\ 0x_1 + x_2 + x_3 = 3 \\ 0x_1 - 4x_2 - 4x_3 = -12 \end{array} \quad \Big(E_3 \mapsto E_3 + 4E_2 \Big) \begin{array}{r} x_1 + 0x_2 + 2x_3 = 5 \\ 0x_1 + x_2 + x_3 = 3 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{array}$$

Thus the original system of 3 equations has the same solution set as the system

$$\begin{aligned} x_1 + 2x_3 &= 5 \\ x_2 + x_3 &= 3. \end{aligned}$$

If we let $x_3 = t$, then $x = (x_1, x_2, x_3)^T$ is a solution when $x_1 = 5 - 2t$, $x_2 = 3 - t$ and $x_3 = t$, that is when $x = (5, 3, 0)^T + t(-2, -1, 1)^T$. Thus the solution set is the line through $p = (5, 3, 0)^T$ in the direction of $u = (-2, -1, 1)^T$.

Matrix Notation

2.6 Remark: We wish to introduce some notation which will simplify our discussion of systems of linear equations. In fact the objects that we introduce will turn out to be of interest in their own right.

2.7 Definition: Let F be a field. An $m\times n$ **matrix** over F is an array of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where each $a_{ij}\in F$. The number a_{ij} is called the (i,j) **entry** of the matrix A, and we write

$$A_{ij}=a_{ij}\,.$$

The set of all $m\times n$ matrices over F is denoted by $M_{m\times n}(F)$. The set of $n\times n$ matrices is also denoted by $M_n(F)$. Note that

$$F^n=M_{n\times 1}(F)\ ,\ M_n(F)=M_{n\times n}(F)\ ,\text{ and } F=M_1(F)\,.$$

The j^{th} **column** of the above matrix $A\in M_{m\times n}(F)$ is the vector $(a_{1j},a_{2j},\cdots,a_{mj})^T\in F^m$. The j^{th} **row** of A is the vector $(a_{j1},a_{j2},\cdots,a_{jn})^T\in F^n$. Given vectors $u_1,u_2,\cdots,u_n\in F^m$, the **matrix with columns** $u_1,u_2,\cdots,u_n$ is the matrix

$$A=\big(u_1,u_2,\cdots,u_n\big)\in M_{m\times n}(F)$$

and given vectors $v_1,v_2,\cdots,v_m\in F^n$, the **matrix with rows** $v_1,v_2,\cdots,v_m$ is the matrix

$$A=\begin{pmatrix} {v_1}^T \\ {v_2}^T \\ \vdots \\ {v_m}^T \end{pmatrix}\in M_{m\times n}(F)\,.$$

Given a matrix $A\in M_{m\times n}(F)$, the **transpose** of A is the matrix $A^T\in M_{n\times m}(F)$ with entries $(A^T)_{ij}=A_{ji}=a_{ji}$, that is

$$A^T=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

The $m\times n$ **zero matrix** is the matrix $0\in M_{m\times n}(F)$ whose entries are all equal to 0. The $n\times n$ **identity** matrix is the matrix $I\in M_n(F)$ with columns $e_1,e_2,\cdots,e_n$. Given a matrix $a=(a_1,a_2,\cdots,a_n)\in M_{1\times n}(F)$ and a vector $x=(x_1,x_2,\cdots,x_n)^T\in F^n$, we define the product $ax\in F$ to be

$$ax=\big(a_1,a_2,\cdots,a_n\big)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\sum_{i=1}^n a_ix_i=a_1x_1+a_2x_2+\cdots+a_nx_n.$$

More generally, given a matrix $A \in M_{m\times n}(F)$ with entries $A_{ij} = a_{ij}$, and a vector $x \in F^n$ with entries $x_i \in F$, we define the product $Ax \in F^m$ to be

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}.$$

Equivalently, we define Ax to be the vector in F^m with entries

$$(Ax)_j = \sum_{i=1}^{n} a_{ji}x_i.$$

Using this notation, notice that for $a = (a_1, a_2, \cdots, a_n) \in M_{1\times n}(F)$ and $b \in F$, the single linear equation $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$ can be written simply as $ax = b$, and for $A \in M_{m\times n}(F)$ with entries $A_{ij} = a_{ij}$ and $b = (b_1, b_2, \cdots, b_n)^T \in F^n$, the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_k \end{aligned}$$

can be written simply as

$$Ax = b.$$

2.8 Note: For vectors $v_1, v_2, \cdots, v_m \in F^n$ and for $x \in F^n$, if A is the matrix with rows $v_1, v_2, \cdots, v_m$ then the product Ax is defined so that we have

$$Ax = \begin{pmatrix} {v_1}^T \\ {v_2}^T \\ \vdots \\ {v_m}^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} {v_1}^T x_1 \\ {v_2}^T x_2 \\ \vdots \\ {v_m}^T x_n \end{pmatrix}.$$

For vectors $u_1, u_2, \cdots, u_n \in F^m$ and $x \in F^n$, if A is the matrix with columns $u_1, u_2, \cdots, u_n$, notice that we have

$$Ax = \big(u_1, u_2, \cdots, u_n\big) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1u_1 + x_2u_2 + \cdots + x_nu_n \in \text{Span}\,\{u_1, u_2, \cdots, u_n\}.$$

2.9 Note: For $0 \in M_{m\times n}(F)$, $I \in M_n(F)$ and $x \in F^n$ we have $0x = 0$ and $Ix = x$.

2.10 Theorem: *(Linearity) For $A \in M_{m\times n}(F)$, we have*
(1) $A(tx) = t\,Ax$ for all $t \in F$ and $x \in F^n$, and
(2) $A(x + y) = Ax + Ay$ for all $x, y \in F^n$.

Proof: For $t \in F$ and for $x, y \in F^n$ we have $\big(A(tx)\big)_j = \sum_{i=1}^{n} A_{ji}(tx)_i = t\sum_{i=1}^{n} A_{ji}x_i = \big(tAx\big)_j$ and $\big(A(x+y)\big)_j = \sum_{i=1}^{n} A_{ji}(x+y)_i = \sum_{i=1}^{n} A_{ji}(x_i + y_i) = \sum_{i=1}^{n} A_{ji}x_i + \sum_{i=1}^{n} A_{ji}y_i = \big(Ax + Ay\big)_j$.

Row Equivalence and Reduced Row Echelon Form

2.11 Definition: Given a matrix $A \in M_{m\times n}(F)$ with entries $A_{ij} = a_{ij}$ and a vector $b \in F^n$, we obtain the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

which we write simply as $Ax = b$. The matrix A is called the **coefficient matrix** of the system, and the matrix $\big(A\,|\,b\big) \in M_{k\times n+1}(F)$ is called the **augmented matrix** of the system. The **solution set** of the equation $Ax = b$ is the set

$$\big\{x \in F^n \big| Ax = b\big\}.$$

2.12 Definition: We noted earlier that we can perform three kinds of operations on the equations in the system without changing the solution set. These correspond to the following three kinds of operations that we can perform on the rows of the augmented matrix without changing the solution set.

(1) $R_i \leftrightarrow R_j$: interchange rows i and j,
(2) $R_i \mapsto t\,R_i$: multiply the i^{th} row by t, where $0 \neq t \in F$, and
(3) $R_i \mapsto R_i + t\,R_j$: add t times the j^{th} row to the i^{th} row, where $t \in F$.

These three kinds of operations will be called **elementary row operations**. Given matrices $A, B \in M_{m\times n}(F)$, we say that A and B are **row equivalent**, and we write $A \sim B$, when B can be obtained by applying a finite sequence of elementary row operations to A.

2.13 Remark: In the next section, we describe an algorithm for finding the solution set to a given matrix equation $Ax = b$. We shall use elementary row operations to construct a sequence of augmented matrices

$$(A|b) = (A_0|b_0) \sim (A_1|b_1) \sim (A_2|b_2) \sim \cdots \sim (A_l, b_l) = (R, c)$$

such that each equation $A_k x = b_k$ has the same solution set as the original equation $Ax = b$, and such that the final matrix $A_l = R$ is in a particularly nice form so that we can easily determine the solution set. We shall find that when the solution set is non-empty, we can write the solutions in the form

$$x = p + t_1u_1 + t_2u_2 + \cdots + t_ru_r.$$

2.14 Definition: A matrix $A \in M_{m\times n}(F)$ is said to be in **reduced row echelon form** when $A = 0$ or there exist column indices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq n$, where $1 \leq r \leq n$, such that for each row index k with $1 \leq k \leq m$ we have

(1) $A_{k,j_k} = 1$,
(2) for each $j < j_k$ we have $A_{kj} = 0$,
(3) for each $i < k$ we have $A_{i,j_k} = 0$, and
(4) for all $i > r$ and all j we have $A_{ij} = 0$.

The entries $A_{k,j_k} = 1$ are called the **pivots**, the positions (k, j_k) where they occur are called the **pivot positions**, the columns $j_1, j_2, \cdots, j_r$ of A are called the **pivot columns**, and the number of pivots r is called the **rank** of the reduced row echelon matrix A.

2.15 Example: Consider the augmented matrix

$$(A|b) = \left(\begin{array}{ccccccc|c} 1 & -2 & 1 & 0 & -3 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Note that the matrix A is in reduced row echelon form with the pivots in positions $(1,1),(2,4)$ and $(3,6)$. The corresponding system of equations is

$$\begin{aligned} x_1 - 2x_2 + x_3 - 3x_5 + 2x_7 &= 4 \\ x_4 + x_5 - x_7 &= 1 \\ x_6 - 2x_7 &= 3 \end{aligned}$$

The solutions x can be obtained by letting $x_2 = t_1$, $x_3 = t_2$, $x_5 = t_3$ and $x_7 = t_4$, with $t_1, t_2, t_3, t_4 \in F$ arbitrary, and then solving for x_1, x_4 and x_6 to get

$$\begin{aligned} x_1 &= 4 + 2t_1 - t_2 + 3t_3 - 2t_4 \\ x_4 &= 1 - t_3 + t_4 \\ x_6 &= 3 + 2t_4 \end{aligned}$$

Thus the solution set is the set of points of the form

$$x = p + t_1u_1 + t_2u_2 + t_3u_3 + t_4u_4$$

where

$$p = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \ u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ u_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ u_4 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

We can also write this as

$$x = p + Bt$$

where B is the matrix with columns u_1, u_2, u_3, u_4.

2.16 Note: In general, suppose that $A \in M_{m\times n}(F)$ is in reduced row echelon form with pivot column indices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq n$. Let $1 \leq l_1 \leq l_2 \leq \cdots \leq l_{n-r} \leq n$ be the non-pivot column indices. Write

$$(A\,|\,b) = \left(\begin{array}{c|c} R & c \\ 0 & d \end{array}\right)$$

where $R \in M_{r\times n}(F)$ is the matrix whose rows are the non-zero rows of A and where $c \in F^r$ and $d \in F^{m-r}$. Then the equation $Ax = b$ has a solution if and only if $d = 0$, and in this case, as in the above example, the solution is given by $x = p + Bt$ where $p \in F^n$ and $B \in M_{n\times m-r}(F)$ are given by $p_J = c$, $p_L = 0$, $B_J = -A_L$, and $B_L = I$ where $p_J = (p_{j_1}, p_{j_2}, \cdots, p_{j_r})^T$, $p_L = (p_{l_1}, p_{l_2}, \cdots, p_{l_{m-r}})^T$, B_J is the matrix whose rows are the rows $j_1, j_2, \cdots, j_r$ of B, B_L is the matrix whose rows are the rows $l_1, l_2, \cdots, l_{m-r}$ of B, and A_L is the matrix whose columns are the columns $l_1, l_2, \cdots, l_{m-r}$ of A.

Gauss-Jordan Elimination

2.17 Theorem: *(Gauss-Jordan Elimination) Let $A \in M_{m\times n}(F)$ and let $b \in F^m$, and consider the equation $Ax = b$. If $A = 0$ and $b = 0$ then every $x \in F^n$ is a solution so the solution set is $\mathbf{F}^n$. If $A = 0$ and $b \neq 0$ then there is no solution, so the solution set is $\emptyset$. If $A \neq 0$ then we can perform a series of elementary row operations to obtain a sequence of augmented matrices*

$$(A|b) = (A_0|b_0) \sim (A_1|b_1) \sim (A_2|b_2) \sim \cdots \sim (A_l|b_l) = \left(\begin{array}{c|c} R & c \\ 0 & d \end{array}\right)$$

where A_l is in reduced row echelon form and R is the matrix whose rows are the non-zero rows of A_l. If $d \neq 0$ then the solution set is empty and if $d = 0$ then the solution set is the affine space $x = p + Bt$, as described in Note 2.16 above.

Proof: Suppose that $A \neq 0$. We describe an algorithm to obtain the required sequence of augmented matrices.

Step 1: choose the smallest index $j = j_1$ such that the j^{th} column of A is not zero, then choose the smallest index $i = i_1$ such that $a_{ij} \neq 0$. Perform the row operations $R_i \mapsto \frac{1}{a_{ij}} R_i$ then $R_1 \leftrightarrow R_i$ to obtain a new augmented matrix $(A'|b')$ whose first $j-1$ columns are all zero, with $A'_{1j} = 1$. For each $i > 1$ perform the row operation $R_i \mapsto R_i - A'_{ij} R_1$ to obtain a new augmented matrix $(A_1|b_1)$ whose first $j-1$ columns are zero and whose j^{th} column is e_1.

Step $s+1$: suppose that we have performed the first s steps in the algorithm and have obtained an augmented matrix (A_s, b_s) which is row-equivalent to $(A|b)$ and which has the property that there exist column indices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_s \leq n$ such that for each row index k with $1 \leq k \leq s$ we have

(1) $(A_s)_{k,j_k} = 1$,
(2) for each $j < j_k$ we have $(A_s)_{kj} = 0$,
(3) for each $i < k$ we have $(A_s)_{i,j_k} = 0$, and
(4) for all $i > s$ and all $j \leq j_s$ we have $(A_s)_{ij} = 0$.

If $s = m$ or if $j_s = n$ then we are done because the matrix A_s is already in reduced row echelon form. Suppose that $s < m$ and $j_s < n$. If for all $i > s$ and all $j > j_s$ we have $(A_s)_{ij} = 0$ then we are done because the matrix A_s is already in row echelon form. Suppose that $A_{ij} \neq 0$ for some $i > s$ and some $j > j_s$. Let $j = j_{s+1}$ be the smallest index such that $(A_s)_{ij} \neq 0$ for some $i > s$, then apply elementary row operations, involving only rows i with $i > s$, to the augmented matrix $(A_s|b_s)$ to obtain a matrix $(A'_s|b'_s)$ with $(A'_s)_{s+1,j} = 1$; this can be done for example by choosing an index $i > s$ such that $(A_s)_{ij} \neq 0$ and then performing the row operation $R_i \mapsto \frac{1}{(A_s)_{ij}} R_i$ then (if $i \neq s+1$) the operation $R_{s+1} \leftrightarrow R_i$. Then for each $i \neq s+1$ perform the row operation $R_i \mapsto R_i - (A'_s)_{ij} R_{s+1}$ to the augmented matrix $(A'_s|b_s)$ to obtain the row-equivalent augmented matrix $(A_{s+1}|b_{s+1})$. Verify that the new matrix satisfies the above 4 properties with s replaced by $s+1$.

2.18 Definition: The algorithm for solving the system $Ax = b$ described in the proof of the above theorem is called **Gauss-Jordan elimination**.

2.19 Example: Solve the system $3x+2y+z=4$, $2x+y-z=3$, $x+2y+4z=1$ in $\mathbf{Q}^3$.

Solution: We form the augmented matrix for the system then perform row operations to reduce to reduced row echelon form.

$$
\begin{aligned}
(A|b) = \left(\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 1 \end{array}\right)
\begin{array}{l} R_1 \mapsto R_1 - R_3 \\ \\ \\ \end{array}
\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 1 \end{array}\right)
\begin{array}{l} \\ R_2 \mapsto R_2 - 2R_1 \\ R_3 \mapsto R_3 - R_1 \end{array} \\
\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right)
\begin{array}{l} R_1 \mapsto R_1 - R_2 \\ \\ R_3 \mapsto R_3 - R_2 \end{array}
\left(\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -3 & 1 \end{array}\right)
\begin{array}{l} \\ \\ R_3 \mapsto -\frac{1}{3}R_3 \end{array} \\
\left(\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)
\begin{array}{l} R_1 \mapsto R_1 - 3R_3 \\ R_2 \mapsto R_2 + 5R_3 \\ \\ \end{array}
\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)
\end{aligned}
$$

From the final reduced matrix we see that the solution is given by $(x,y,z) = \left(1, \frac{2}{3}, -\frac{1}{3}\right)$.

2.20 Remark: Note that in the above solution, at the first step we used the row operation $R_1 \mapsto R_1 - R_3$ to obtain a pivot in position $(1,1)$, but we could have achieved this in many different ways. For example, we could have used the row operation $R_1 \leftrightarrow R_3$ or we could have used $R_1 \mapsto \frac{1}{3}R_1$.

2.21 Example: Solve the system $2x+y+3z=1$, $3x+y+5z=2$, $x-y+3z=0$ in $\mathbf{Q}^3$.

Solution: Using Gauss-Jordan elimination, we have

$$
\begin{aligned}
(A|b) &= \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 3 & 1 & 5 & 2 \\ 1 & -1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 3 & 1 & 5 & 2 \\ 1 & -1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right)
\end{aligned}
$$

From the reduced matrix, we see that there is no solution.

2.22 Example: Solve the system $x_1+2x_2+x_3+3x_4=2$, $2x_1+3x_2+x_3+4x_4+x_5=5$, $x_1+3x_2+2x_3+4x_4+x_5=3$ in $\mathbf{Q}^5$.

Solution: Using Gauss-Jordan elimination gives

$$
\begin{aligned}
(A|b) &= \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 2 \\ 2 & 3 & 1 & 4 & 1 & 5 \\ 1 & 3 & 2 & 4 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 2 & 4 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 & -3 & -2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & -3 & -2 \end{array}\right)
\end{aligned}
$$

From the reduced matrix we see that the solution set is the plane given by $x = p + su + tv$ where $p = (2,3,0,-2,0)^T$, $u = (1,-1,1,0,0)^T$ and $v = (1,-5,0,3,1)^T$.