MATH 147 Calculus 1, Lecture Notes by Stephen New

Chapter 4: Limits of Functions

4.1 Definition: Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, and let $f : A \to F$. For $a \in F$, we say that a is a **limit point** of A when

$$\forall \delta > 0 \; \exists x \in A \; 0 < |x - a| \leq \delta.$$

When a is a limit point of A, we make the following definitions.

(1) For $b \in F$, we say that the **limit** of $f(x)$ as x tends to a is equal to b, and we write $\lim\limits_{x \to a} f(x) = b$ or we write $f(x) \to b$ as $x \to a$, when

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in A \; \big(0 < |x - a| \leq \delta \Longrightarrow |f(x) - b| \leq \epsilon\big).$$

(2) We say the limit of $f(x)$ as x tends to a is equal to **infinity**, and we write $\lim\limits_{x \to a} f(x) = \infty$, or we write $f(x) \to \infty$ as $x \to a$, when

$$\forall r \in F \; \exists \delta > 0 \; \forall x \in A \; \big(0 < |x - a| \leq \delta \Longrightarrow f(x) \geq r\big).$$

(3) We say that the limit of $f(x)$ as x tends to a is equal to **negative infinity**, and we write $\lim\limits_{x \to a} f(x) = -\infty$, or we write $f(x) \to -\infty$ as $x \to a$, when

$$\forall r \in F \; \exists \delta > 0 \; \forall x \in A \; \big(0 < |x - a| \leq \delta \Longrightarrow f(x) \leq r\big).$$

For $a \in F$, we say that a is a **limit point of A from below** when

$$\forall \delta > 0 \; \exists x \in A \quad a - \delta \leq x < a\,.$$

When a is a limit point of A from below and $b \in F$, we make the following definitions.

(4) $\lim\limits_{x \to a^-} f(x) = b \iff \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in A \; \big(a - \delta \leq x < a \Longrightarrow |f(x) - b| \leq \epsilon\big).$

(5) $\lim\limits_{x \to a^-} f(x) = \infty \iff \forall r \in F \; \exists \delta > 0 \; \forall x \in A \; \big(a - \delta \leq x < a \Longrightarrow f(x) \geq r\big).$

(6) $\lim\limits_{x \to a^-} f(x) = -\infty \iff \forall r \in F \; \exists \delta > 0 \; \forall x \in A \; \big(a - \delta \leq x < a \Longrightarrow f(x) \leq r\big).$

For $a \in F$, we say that a is a **limit point of A from above** when

$$\forall \delta > 0 \; \exists x \in A \quad a < x \leq a + \delta\,.$$

When a is a limit point of A from above and $b \in F$, we make the following definitions.

(7) $\lim\limits_{x \to a^+} f(x) = b \iff \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in A \; \big(a < x \leq a + \delta \Longrightarrow |f(x) - b| \leq \epsilon\big).$

(8) $\lim\limits_{x \to a^+} f(x) = \infty \iff \forall r \in F \; \exists \delta > 0 \; \forall x \in A \; \big(a < x \leq a + \delta \Longrightarrow f(x) \geq r\big).$

(9) $\lim\limits_{x \to a^+} f(x) = -\infty \iff \forall r \in F \; \exists \delta > 0 \; \forall x \in A \; \big(a < x \leq a + \delta \Longrightarrow f(x) \leq r\big).$

We say that infinity is a limit point of A (from below) when A is not bounded above, that is when $\forall m\in F\ \exists x\in A\ \ x\geq m$. When A is not bounded above and $b\in F$, we make the following definitions.

(10) $\lim\limits_{x\to\infty} f(x)=b \iff \forall\epsilon>0\ \exists m\in F\ \forall x\in A\ \big(x\geq m\Longrightarrow |f(x)-b|\leq\epsilon\big).$

(11) $\lim\limits_{x\to\infty} f(x)=\infty \iff \forall r\in F\ \exists m\in F\ \forall x\in A\ \big(x\geq m\Longrightarrow f(x)\geq r\big).$

(12) $\lim\limits_{x\to\infty} f(x)=-\infty \iff \forall r\in F\ \exists m\in F\ \forall x\in A\ \big(x\geq m\Longrightarrow f(x)\leq r\big).$

We say that negative infinity is a limit point of A (from above) when A is not bounded below, that is when $\forall m\in F\ \exists x\in A\ \ x\leq m$. When A is not bounded below and $b\in F$, we make the following definitions.

(13) $\lim\limits_{x\to-\infty} f(x)=b \iff \forall\epsilon>0\ \exists m\in F\ \forall x\in A\ \big(x\leq m\Longrightarrow |f(x)-b|\leq\epsilon\big).$

(14) $\lim\limits_{x\to-\infty} f(x)=\infty \iff \forall r\in F\ \exists m\in F\ \forall x\in A\ \big(x\leq m\Longrightarrow f(x)\geq r\big).$

(15) $\lim\limits_{x\to-\infty} f(x)=-\infty \iff \forall r\in F\ \exists m\in F\ \forall x\in A\ \big(x\leq m\Longrightarrow f(x)\leq r\big).$

4.2 Example: Let $f(x)=\dfrac{x^2+2x-3}{x^2-1}$. Show that $\lim\limits_{x\to 1} f(x)=2$.

Solution: Note that for $x\neq 1$ we have

$$|f(x)-2|=\left|\tfrac{x^2+2x-3}{x^2-1}-2\right|=\left|\tfrac{(x+3)(x-1)}{(x+1)(x-1)}-2\right|=\left|\tfrac{x+3}{x+1}-2\right|=\left|\tfrac{x+3-2x-2}{x+1}\right|=\left|\tfrac{-x+1}{x+1}\right|=\tfrac{|x-1|}{|x+1|}.$$

Let $\epsilon>0$. Choose $\delta=\min\{1,\epsilon\}$. Let $0<|x-1|\leq\delta$. Since $0<|x-1|$ we have $x\neq 1$ so, as shown above, $|f(x)-2|=\frac{|x-1|}{|x+1|}$. Since $|x-1|\leq\delta\leq 1$ we have $0\leq x\leq 3$ so that $1\leq x+1$, and hence $|f(x)-2|=\frac{|x-1|}{|x+1|}\leq|x-1|$. Finally, since $|x-a|\leq\delta\leq\epsilon$ we have $|f(x)-2|\leq|x-1|\leq\epsilon$. Thus $\lim\limits_{x\to 1} f(x)=2$.

4.3 Theorem: *(Two Sided Limits) Let F be a subfield of $\mathbf{R}$, let $A\subseteq F$, let $f:A\to F$ and let $a\in F$. Suppose that a is a limit point of A both from the left and from the right. Then for $u\in\hat{F}$ we have $\lim\limits_{x\to a} f(x)=a$ if and only if $\lim\limits_{x\to a^-} f(x)=u=\lim\limits_{x\to a^+} f(x)$.*

Proof: We prove the theorem in the case that $u=b\in F$. Suppose that $\lim\limits_{x\to a} f(x)=b\in F$. Let $\epsilon>0$. Choose $\delta>0$ so that for all $x\in A$, if $0<|x-a|\leq\delta$ then $|f(x)-b|\leq\epsilon$. For $x\in A$ with $a-\delta\leq x<a$ we have $0<|x-a|\leq\delta$ and so $|f(x)-b|\leq\epsilon$. This shows that $\lim\limits_{x\to a^-} f(x)=b$. For $x\in A$ with $a<x\leq x+\delta$ we have $0<|x-a|\leq\delta$ and so $|f(x)-b|\leq\epsilon$. This show that $\lim\limits_{x\to a^+} f(x)=b$.

Conversely, suppose that $\lim\limits_{x\to a^-} f(x)=b=\lim\limits_{x\to a^+} f(x)$. Let $\epsilon>0$. Since $f(x)\to b$ as $x\to a^-$, we can choose $\delta_1>0$ so that for all $x\in A$ with $a-\delta\leq a<a$ we have $|f(x)-b|\leq\epsilon$. Since $f(x)\to b$ as $x\to a^+$ we can choose $\delta_2>0$ so that for all $x\in A$ with $a<x\leq a+\delta_2$ we have $|f(x)-b|\leq\epsilon$. Let $\delta=\min\{\delta_1,\delta_2\}$. Let $x\in A$ with $0<|x-a|\leq\delta$. Either we have $x<a$ or we have $x>a$. In the case that $x<a$ we have $a-\delta_1\leq a-\delta\leq x<a$ and so $|f(x)-b|\leq\epsilon$ (by the choice of δ_1). In the case that $x>a$ we have $a<x\leq a+\delta\leq a+\delta_2$ and so $|f(x)-b|\leq\epsilon$ (by the choice of δ_2. In either case we have $|f(x)-b|\leq\epsilon$, and so $\lim\limits_{x\to a} f(x)=b$, as required.

4.4 Remark: For the sequence $\langle x_k \rangle_{k\geq p}$ in F given by $x_k = f(k)$ where $f : \mathbf{Z}_{\geq p} \to F$, the definitions (10), (11) and (12) agree with our definitions for limits of sequences. Thus limits of sequences are a special case of limits of functions. The following theorem shows that limits of functions are determined by limits of sequences.

4.5 Theorem: *(The Sequential Characterization of Limits of Functions) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$, and let $u \in \hat{F}$.*
(1) When $a \in F$ is a limit point of A, $\lim\limits_{x\to a} f(x) = u$ if and only if for every sequence $\langle x_k \rangle$ in $A \setminus \{a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(2) When a is a limit point of A from below, $\lim\limits_{x\to a^-} f(x) = u$ if and only if for every sequence $\langle x_k \rangle$ in $\{x \in A | x < a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(3) When a is a limit point of A from above, $\lim\limits_{x\to a^+} f(x) = u$ if and only if for every sequence $\langle x_k \rangle$ in $\{x \in A | x > a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(4) When A is not bounded above, $\lim\limits_{x\to \infty} f(x) = u$ if and only if for every sequence $\langle x_k \rangle$ in A with $x_k \to \infty$ we have $f(x_k) \to u$.
(5) When A is not bounded below, $\lim\limits_{x\to -\infty} f(x) = u$ if and only if for every sequence $\langle x_k \rangle$ in A with $x_k \to -\infty$ we have $f(x_k) \to u$.

Proof: We prove Part (1) in the case that $u = b \in F$. Let $a \in F$ be a limit point of A. Suppose that $\lim\limits_{x\to a} f(x) = b \in F$. Let $\langle x_k \rangle$ be a sequence in $A \setminus \{a\}$ with $x_k \to a$. Let $\epsilon > 0$. Since $\lim\limits_{x\to a} f(x) = b$, we can choose $\delta > 0$ so that $0 < |x - a| \leq \delta \Longrightarrow |f(x) = b| \leq \epsilon$. Since $x_k \to a$ we can choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |x_k - a| \leq \delta$. Then for $k \geq m$, we have $|x_k - a| \leq \delta$ and we have $x_k \neq a$ (since the sequence $\langle x_k \rangle$ is in the set $A \setminus \{a\}$) so that $0 < |x - a| \leq \delta$ and hence $|f(x_k) - b| \leq \epsilon$. This shows that $f(x_k) \to b$.

Conversely, suppose that $\lim\limits_{x\to a} f(x) \neq b$. Choose $\epsilon_0 > 0$ so that for all $\delta > 0$ there exists $x \in A$ with $0 < |x - a| \leq \delta$ and $|f(x) - b| > \epsilon_0$. For each $k \in \mathbf{Z}^+$, choose $x_k \in A$ with $0 < |x_k - a| \leq \frac{1}{k}$ and $|f(x_k) - b| > \epsilon_0$. In this way we obtain a sequence $\langle x_k \rangle_{k\geq 1}$ in $A \setminus \{a\}$ (we remark that the Axiom of Choice is required to construct this sequence $\langle x_k \rangle$). Since $|x_k - a| \leq \frac{1}{k}$ for all $k \in \mathbf{Z}^+$, it follows that $x_k \to a$ (indeed, given $\epsilon > 0$ we can choose $m \in \mathbf{Z}$ with $m \geq \frac{1}{\epsilon}$ and then $k \geq m \Longrightarrow |x_k - a| \leq \frac{1}{k} \leq \frac{1}{m} \leq \epsilon$). Since $|f(x_k) - b| > \epsilon_0$ for all k, it follows that $f(x_k) \not\to b$ (indeed if we had $f(x_k) \to b$ we could choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |f(x_k) - b| \leq \epsilon_0$ and then we could choose $k = m$ to get $|f(x_k) - b| \leq \epsilon_0$).

4.6 Remark: It follows from the Sequential Characterization of Limits of Functions that all of our theorems about limits of sequences imply analogous theorems in the more general setting of limits of functions. We list several of those theorems and give one sample proof.

4.7 Theorem: *(Local Determination of Limits) Let F be a subfield of $\mathbf{R}$, let $A, B \subseteq F$, let $f : A \to F$ and let $g : B \to F$. Suppose that $a \in F$ is a limit point of both sets A and B, and that for some $\delta > 0$ we have $C = \big\{x \in A \big| 0 < |x - a| \leq \delta\big\} \subseteq \big\{x \in B \big| 0 < |x - a| \leq \delta\big\}$ and that $f(x) = g(x)$ for all $x \in C$. Then if $\lim\limits_{x\to a} g(x) = u \in \hat{F}$ then $\lim\limits_{x\to a} f(x) = u$.*
Similar results holds for limits $x \to a^\pm$ and $x \to \pm\infty$.

4.8 Theorem: *(Uniqueness of Limits) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$, and let a be a limit point of A. For $u, v \in \hat{F}$, if $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} f(x) = v$ then $u = v$.*
Similar results hold for limits $x \to a^\pm$ and $x \to \pm\infty$.

4.9 Theorem: *(Extended Operations on Limits) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f, g : A \to F$ and let a be a limit point of A. Let $u, v \in \hat{F}$ and suppose that $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} g(x) = v$. Then*

(1) if $u + v$ is defined in $\hat{F}$ then $\lim\limits_{x\to a}(f+g)(x) = u + v$,

(2) if $u - v$ is defined in $\hat{F}$ then $\lim\limits_{x\to a}(f-g)(x) = u - v$,

(3) if $u \cdot v$ is defined in $\hat{F}$ then $\lim\limits_{x\to a}(fg)(x) = u \cdot v$, and

(4) if u/v is defined in $\hat{F}$ then $\lim\limits_{x\to a}(f/g)(x) = u/v$.

Similar results hold for limits $x \to a^{\pm}$ and $x \to \pm\infty$.

Proof: We prove Part (4). Suppose that u/v is defined in $\hat{F}$. Let $\langle x_k \rangle$ be any sequence in $A \setminus \{a\}$ with $x_k \to a$. By the Sequential Characterization of Limits, since $\lim\limits_{x\to a} f(x) = u$ we have $f(x_k) \to u$, and since $\lim\limits_{x\to a} g(x) = v$ we have $f(x_k) \to v$. By Extended Operations on Limits of Sequences (Theorem 3.13), since $f(x_k) \to u$ and $g(x_k) \to v$ and u/v is defined in $\hat{F}$, we have $(f/g)(x_k) = \frac{f(x_k)}{g(x_k)} \to u/v$. Thus $(f/g)(x_k) \to u/v$ for every sequence $\langle x_k \rangle$ in $A \setminus \{a\}$ with $x_k \to a$. By the Sequential Characterization of Limits, it follows that $\lim\limits_{x\to a}(f/g)(x) = u/v$.

4.10 Theorem: *(Basic Limits) Let F be a subfield of $\mathbf{R}$, and let $A \subseteq F$. For the constant function $f : A \to F$ given by $f(x) = b$ for all $x \in A$, we have*

$$\lim_{x\to a} f(x) = b\ ,\ \lim_{x\to a^+} f(x) = b\ ,\ \lim_{x\to a^-} f(x) = b\ ,\ \lim_{x\to\infty} f(x) = b \text{ and } \lim_{x\to -\infty} f(x) = b,$$

and for the identity function $f : A \to F$ given by $f(x) = x$ for all $x \in A$ we have

$$\lim_{x\to a} f(x) = a\ ,\ \lim_{x\to a^+} f(x) = a\ ,\ \lim_{x\to a^-} f(x) = a\ ,\ \lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to -\infty} f(x) = -\infty$$

whenever the limits are defined.

4.11 Theorem: *(Basic Elementary Functions Acting on Limits) For $f : A \subseteq \mathbf{R} \to \mathbf{R}$ and for $a, b, c \in \mathbf{R}$ with a a limit point of A, we have the following.*

(1) If $\lim\limits_{x\to a} f(x) = b > 0$ then $\lim\limits_{x\to a} f(x)^c = b^c$,

$$\text{if } \lim_{x\to a} f(x) = \infty \text{ then } \lim_{x\to a} f(x)^c = \begin{cases} \infty \text{ if } c > 0 \\ 1 \text{ if } c = 0 \\ 0 \text{ if } c < 0, \end{cases}$$

$$\text{if } f(x) > 0 \text{ for all } x \in A \text{ and } \lim_{x\to a} f(x) = 0 \text{ then } \lim_{x\to a} f(x)^c = \begin{cases} 0 \text{ if } c > 0 \\ 1 \text{ if } c = 0 \\ \infty \text{ if } c < 0. \end{cases}$$

(2) If $\lim\limits_{x\to a} f(x) = b$ and $c > 0$ then $\lim\limits_{x\to a} c^{f(x)} = c^b$,

$$\text{if } \lim_{x\to a} f(x) = \infty \text{ and } c > 0 \text{ then } \lim_{x\to \infty} c^{f(x)} = \begin{cases} \infty \text{ if } c > 1 \\ 1 \text{ if } c = 1 \\ 0 \text{ if } 0 < c < 1, \end{cases}$$

$$\text{if } \lim_{x\to a} f(x) = -\infty \text{ and } c > 0 \text{ then } \lim_{x\to a} c^{f(x)} = \begin{cases} 0 \text{ if } c > 1 \\ 1 \text{ if } c = 1 \\ 0 \text{ if } 0 < c < 1. \end{cases}$$

(3) If $\lim\limits_{x\to a} f(x) = b > 0$ *and* $c > 0$ *then* $\lim\limits_{x\to a} \log_c f(x) = \log_c b$,

if $\lim\limits_{x\to a} f(x) = \infty$ *and* $c > 0$ *then* $\lim\limits_{x\to a} \log_c f(x) = \begin{cases} \infty \text{ if } c > 1 \\ -\infty \text{ if } 0 < c < 1, \end{cases}$

if $f(x) > 0$ *for all* $x \in A$, $\lim\limits_{x\to a} f(x) = 0$ *and* $c > 0$ *then* $\lim\limits_{x\to a} \log_c f(x) = \begin{cases} -\infty \text{ if } c > 1 \\ \infty \text{ if } 0 < c < 1. \end{cases}$

(4) If $\lim\limits_{x\to a} f(x) = b$ *then* $\lim\limits_{x\to a} \sin f(x) = \sin b$ *and* $\lim\limits_{x\to a} \cos f(x) = \cos b$,
the limits $\lim\limits_{x\to\pm\infty} \sin x$, $\lim\limits_{\pm\infty} \cos x$ *and* $\lim\limits_{x\to\pm\infty} \tan x$ *do not exist.*

(5) If $f(x) \in [-1,1]$ *for all* $x \in A$ *and* $\lim\limits_{x\to a} f(x) = b$ *then* $\lim\limits_{x\to a} \sin^{-1} f(x) = \sin^{-1} b$,
if $\lim\limits_{x\to a} f(x) = b \in \mathbf{R}$ *then* $\lim\limits_{x\to a} \tan^{-1} f(x) = \tan^{-1} b$,
if $\lim\limits_{x\to a} f(x) = \infty$ *then* $\lim\limits_{x\to a} \tan^{-1} f(x) = \frac{\pi}{2}$, *and*
if $\lim\limits_{x\to a} f(x) = -\infty$ *then* $\lim\limits_{x\to a} \tan^{-1} f(x) = -\frac{\pi}{2}$.

Similar results hold for limits $x \to a^{\pm}$ *and* $x \to \pm\infty$.

4.12 Example: Evaluate each of the following limits, if they exist.

(a) $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{3-x}$

(b) $\lim\limits_{x\to 1} \sin^{-1}\left(\dfrac{2}{x-1} - \dfrac{x+3}{x^2-1}\right)$

(c) $\lim\limits_{x\to 0} e^{-1/x^2}$

(d) $\lim\limits_{x\to\infty} \dfrac{(3x+1)\sqrt{x}}{\sqrt{4x^3-2x+1}}$

(e) $\lim\limits_{x\to 1^-} \dfrac{\sqrt{x^3-2x^2+x}}{x^2+2x-3}$

(f) $\lim\limits_{x\to -1^+} \dfrac{x^2-2x-3}{x^3+4x^2+5x+2}$

Solution: I may include solutions later.

4.13 Theorem: *(The Comparison Theorem) Let* F *be a subfield of* $\mathbf{R}$, *let* $A \subseteq F$, *let* f *and* g *be two functions* $f, g : A \to F$ *and let* $a \in F$ *be a limit point of* A. *Suppose that* $f(x) \leq g(x)$ *for all* $x \in A$. *Then*

(1) if $\lim\limits_{x\to a} f(x) = u$ *and* $\lim\limits_{x\to a} f(x) = v$ *with* $u, v \in \hat{F}$, *then* $u \leq v$,
(2) if $\lim\limits_{x\to a} f(x) = \infty$ *then* $\lim\limits_{x\to a} g(x) = \infty$, *and*
(3) if $\lim\limits_{x\to a} g(x) = -\infty$ *then* $\lim\limits_{x\to a} g(x) = -\infty$.

Similar results hold for limits $x \to a^{\pm}$ *and* $x \to \pm\infty$.

4.14 Theorem: *(The Squeeze Theorem) Let* F *be a subfield of* $\mathbf{R}$, *let* $A \subseteq F$, *let* $f, g, h : A \to F$, *and let* a *be a limit point of* A.

(1) If $f(x) \leq g(x) \leq h(x)$ *for all* $x \in A$ *and* $\lim\limits_{x\to a} f(x) = b = \lim\limits_{x\to a} h(x)$, *then* $\lim\limits_{x\to a} g(x) = b$.
(2) If $|f(x)| \leq g(x)$ *for all* $x \in A$ *and* $\lim\limits_{x\to a} g(x) = 0$ *then* $\lim\limits_{x\to a} f(x) = 0$.

Similar results hold for limits $x \to a^{\pm}$ *and* $x \to \pm\infty$.

4.15 Definition: Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, and let $f : A \to F$. For $a \in A$, we say that f is **continuous** at a when

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall x \in A \ \ \big(|x - a| \leq \delta \Longrightarrow |f(x) - f(a)| \leq \epsilon\big).$$

We say that f is **continuous** (in A) when f is continuous at every point $a \in A$.

4.16 Theorem: *Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$ and let $a \in A$. Then*
(1) if a is not a limit point of A then f is continuous at a, and
(2) if a is a limit point of A then f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

Proof: The proof is left as an exercise.

4.17 Theorem: *(The Sequential Characterization of Continuity) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f : A \to F$ and let $a \in A$. Then f is continuos at a if and only if for every sequence $\langle x_k \rangle$ in A with $x_k \to a$ we have $f(x_k) \to f(a)$.*

Proof: Suppose that f is continuous at a. Let $\langle x_k \rangle$ be a sequence in A with $x_k \to a$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for all $x \in A$ we have $|x - a| \leq \delta \Longrightarrow |f(x) - f(a)| \leq \epsilon$. Choose $m \in \mathbf{Z}$ so that for all indices k we have $k \geq m \Longrightarrow |x_k - a| \leq \delta$. Then when $k \geq m$ we have $|x_k - a| \leq \delta$ and hence $|f(x_k) - f(a)| \leq \epsilon$. Thus we have $f(x_k) \to f(a)$.

Conversely, suppose that f is not continuous at a. Choose $\epsilon_0 > 0$ so that for all $\delta > 0$ there exists $x \in A$ with $|x - a| \leq \delta$ and $|f(x) - f(a)| > \epsilon_0$. For each $k \in \mathbf{Z}^+$, choose $x_k \in A$ with $|x_k - a| \leq \frac{1}{k}$ and $|f(x_k) - f(a)| > \epsilon_0$. Consider the sequence $\langle x_k \rangle$ in A (we remark that the Axiom of Choice is being used here). Since $|x_k - a| \leq \frac{1}{k}$ for all $k \in \mathbf{Z}^+$, it follows that $x_k \to a$. Since $|f(x_k) - f(a)| > \epsilon_0$ for all $k \in \mathbf{Z}^+$, it follows that $f(x_k) \not\to f(a)$.

4.18 Theorem: *(Operations on Continuous Functions) Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, let $f, g : A \to F$, let $a \in A$ and let $c \in F$. Suppose that f and g are continuous at a. Then the functions cf, $f + g$, $f - g$ and fg are all continuous at a, and if $g(a) \neq 0$ then the function f/g is continuous at a.*

Proof: The proof is left as an exercise.

4.19 Theorem: *(Composition of Continuous Functions) Let F be a subfield of $\mathbf{R}$, let $A, B \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, let $h = g \circ f : C \to \mathbf{R}$ where $C = A \cap f^{-1}(B)$.*
(1) If f is continuous at $a \in C$ and g is continuous at $f(a)$, then h is continuous at a.
(2) If f is continuous (in A) and g is continuous (in B) then h is continuous (in C).

Proof: Note that Part (2) follows immediately from Part (1), so it suffices to prove Part (1). Suppose that f is continuous at $a \in A$ and g is continuous at $b = f(a) \in B$. Let $\langle x_k \rangle$ be a sequence in C with $x_k \to a$. Since f is continuous at a, we have $f(x_k) \to f(a) = b$ by the Sequential Characterization of Continuity. Since $\big\langle f(x_k) \big\rangle$ is a sequence in B with $f(x_k) \to b$ and since g is continuous at b, we have $g(f(x_k)) \to g(b)$ by the Sequential Characterization of Continuity. Thus we have $h(x_k) = g\big(f(x_k)\big) \to g(b) = g\big(f(a)\big) = h(a)$. We have shown that for every sequence $\langle x_k \rangle$ in C with $x_k \to a$ we have $h(x_k) \to h(a)$. Thus h is continuous at a by the Sequential Characterization of Continuity.

4.20 Corollary: *Every elementary function is continuous (in its domain).*

Proof: The basic elementary functions are all continuous in their domains by the Basic Elementary Functions Acting on Limits Theorem. If follows that every elementary function is continuous by Theorems 4.18 and 4.19.

4.21 Theorem: *(Functions Acting on Limits) Let F be a subfield of $\mathbf{R}$, let $A, B \subseteq F$, let $f : A \to F$, let $g : B \to F$ and let $h = g \circ f : C \to F$ where $C = A \cap f^{-1}(B)$. Let a be a limit point of C (hence also of A) and let b be a limit point of B. Suppose that $\lim_{x \to a} f(x) = a$ and $\lim_{y \to b} g(y) = c$. Suppose either that $f(x) \neq b$ for all $x \in C \setminus \{a\}$ or that g is continuous at $b \in B$. Then $\lim_{x \to a} h(x) = c$.*

Analogous results hold, dealing with limits $x \to a^{\pm}$, $x \to \pm\infty$, $y \to b^{\pm}$ and $y \to \pm\infty$.

Proof: The proof is similar to the proof of the Composition of Continuous Functions Theorem.

4.22 Theorem: *(Intermediate Value Theorem) Let I be an interval in $\mathbf{R}$ and let $f : I \to \mathbf{R}$ be continuous. Let $a, b \in I$ with $a \leq b$ and let $y \in \mathbf{R}$. Suppose that either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$. Then there exists $x \in [a, b]$ with $f(x) = y$.*

Proof: We prove the theorem in the case that $f(a) \leq y \leq f(b)$. If $y = f(a)$ then we can take $x = a$ and if $y = f(b)$ then we can take $x = b$. Suppose that $f(a) < y < f(b)$. Let $A = \{t \in [a, b] \big| f(t) \leq y\}$. Note that $A \neq \emptyset$ (since $a \in A$) and A is bounded above (by b) and so A has a supremum in $\mathbf{R}$. Let $x = \sup A$. Since $a \in A$ and $x = \sup A$ we have $x \geq a$. Since b is an upper bound for A and $x = \sup A$ we have $x \leq b$. Thus $x \in [a, b]$.

We claim that $f(x) = y$. Suppose, for a contradiction, that $f(x) > y$. Since $x \neq a$ (because $f(a) < y$ but $f(x) > y$) we can choose $\delta_1 > 0$ so that $[x - \delta_1, x] \subseteq [a, b]$. Since f is continuous at x with $f(x) > y$, we can choose δ_2 so that for all $t \in [a, b]$ we have $|t - x| \leq \delta_2 \Longrightarrow f(t) > y$. Let $\delta = \min\{\delta_1, \delta_2\}$. Since $x = \sup A$, by the Approximation Property we can choose $t \in A$ with $x - \delta \leq t \leq x$. Since $t \in A$ we have $f(t) \leq y$, but since $t \in [x - \delta, x]$ we have $f(t) > y$, so we have obtained the desired contradiction. Now suppose, for a contradiction, that $f(x) < y$. Since $x \neq b$ (because $f(b) > y$ but $f(x) < y$) we can choose $\delta_1 > 0$ so that $[x, x + \delta_1] \subseteq [a, b]$. Since f is continuous at x with $f(x) < y$ we can choose $\delta_2 > 0$ so that for all $t \in [a, b]$ we have $|t - x| \leq \delta_2 \Longrightarrow f(t) < y$. Let $\delta = \min\{\delta_1, \delta_2\}$ so that $[x, x + \delta] \subseteq [a, b]$ and for all $t \in [x, x + \delta]$ we have $f(t) < y$. But then $x + \delta \in A$ so we cannot have $x = \sup A$, and we have obtained the desired contradiction.

4.23 Example: Define $f : \mathbf{Q} \to \mathbf{Q}$ be $f(x) = x^2$. For $a = 0$ and $b = 2$ and $y = 2$ we have $f(a) < y < f(b)$ but there is no point x in the rational interval $[a, b] = \{t \in \mathbf{Q} | a \leq t \leq b\}$ for which $f(x) = y$. So the conclusion of the Intermediate Value Theorem does not hold in this case.

4.24 Definition: Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, and let $f : A \to F$. For $a \in A$, if $f(a) \geq f(x)$ for every $x \in A$, then we say that $f(a)$ is the **maximum value** of f and that f attains its maximum value at a. Similarly for $b \in A$, if $f(b) \leq f(x)$ for every $x \in A$ then we say that $f(b)$ is the **minimum value** of f (in A) and that f attains its minimum value at b. We say that f attains its **extreme values** in A when f attains its maximum value at some point $a \in A$ and f attains its minimum value at some point $b \in A$.

4.25 Theorem: *(Extreme Value Theorem) Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a, b] \to \mathbf{R}$ be continuous. Then f attains its extreme values in $[a, b]$.*

Proof: We prove that f attains its maximum. First we claim that f is bounded above. Suppose, for a contradiction, that it is not. For each $k \in \mathbf{Z}^+$, choose $x_k \in [a, b]$ such that $f(x_k) \geq k$. By the Bolzano Weierstrass Theorem, we can choose a convergent subsequence $\langle x_{k_j} \rangle$. Let $p = \lim_{j\to\infty} x_{k_j}$. Note that $p \in [a, b]$ by Comparison (since $x_{k_j} \geq a$ for all j we have $p \geq a$, and since $x_{k_j} \leq b$ for all j we have $p \leq b$). Since $f(x_{k_j}) \geq k_j$ and $k_j \to \infty$ we must have $f(x_{k_j}) \to \infty$ as $j \to \infty$. But by the Sequential Characterization of Continuity, we should have $f(x_{k_j}) \to f(p) \in \mathbf{R}$, so we have obtained the desired contradiction. Thus f is bounded above, as claimed.

Since the range $f([a, b])$ is nonempty and bounded above, it has a supremum. Let $m = \sup f([a, b])$. By the Approximation Property of the supremum, for each $k \in \mathbf{Z}^+$ we can choose $y_k \in [a, b]$ such that $m - \frac{1}{k} \leq f(y_k) \leq m$. By the Bolzano Weierstrass Theorem, we can choose a convergent subsequence $\langle y_{k_j} \rangle$. Let $c = \lim_{j\to\infty} y_{k_j}$. Since we have $m - \frac{1}{k_j} \leq f(y_{k_j}) \leq m$ and $\frac{1}{k_j} \to 0$, we have $f(y_{k_j}) \to m$ as $j \to \infty$ by the Squeeze Theorem. Since f is continuous at c, by the Sequential Characterization of Continuity we have $f(y_{k_j}) \to f(c)$, and so by the Uniqueness of Limits, we have $f(c) = m$. Thus f attains its maximum value at c.

4.26 Example: For the function $f : [-1, 1] \subseteq \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3 - x$, you can check using high school calculus that f attains its maximum and minimum values at $a = -\frac{1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{3}}$. The function $f : [-1, 1] \subseteq \mathbf{Q} \to \mathbf{Q}$ is continuous in the closed rational interval $[-1, 1] = \{t \in \mathbf{Q} | -1 \leq t \leq 1\}$, but it does not attain its maximum and minimum values in this interval, so the conclusion of the Extreme Value Theorem does not hold for this function.

4.27 Definition: Let F be a subfield of $\mathbf{R}$, let $A \subseteq F$, and let $f : A \to F$. We say that f is **uniformly continuous** in A when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall a \in A\ \forall x \in A \big(|x - a| \leq \delta \Longrightarrow |f(x) - f(a)| \leq \epsilon \big).$$

4.28 Example: Define $f : (0, \infty) \to (0, \infty)$ by $f(x) = \frac{1}{x}$. Let $\epsilon = 1$. Let $\delta > 0$. If $\delta \geq 1$ then for $x = \frac{1}{3}$ and $a = 1$ we have $|x - a| = \frac{2}{3} \leq \delta$ but $|f(x) - f(a)| = 2 > \epsilon$. If $0 < \delta < 1$ then for $x = \frac{\delta}{3}$ and $a = \delta$ we have $|x - a| = \frac{2}{3}\delta \leq \delta$ but $|f(x) - f(a)| = \frac{2}{\delta} \geq 2 > \epsilon$. This proves that f is not uniformly continuous (but f is continuous because it is elementary).

4.29 Theorem: *(Closed Bounded Intervals and Uniform Continuity) Let $a, b \in \mathbf{R}$ with $a < b$ and let $f : [a, b] \to \mathbf{R}$. If f is continuous then f is uniformly continuous (on $[a, b]$).*

Proof: Suppose, for a contradiction, that $f : [a, b] \to \mathbf{R}$ is continuous but not uniformly continuous on $[a, b]$. Choose $\epsilon > 0$ so that for all $\delta > 0$ there exist $x, y \in [a, b]$ such that $|x - y| \leq \delta$ but $|f(x) - f(y)| > \epsilon$. For each $k \in \mathbf{Z}^+$ choose x_k and y_k in $[a, b]$ with $|x_k - y_k| \leq \frac{1}{k}$ and $|f(x_k) - f(y_k)| > \epsilon$. By the Bolzano Weierstrass Theorem, we can choose a convergent subsequence $\langle y_{k_j} \rangle$ of $\langle y_k \rangle$. Let $c = \lim_{j\to\infty} y_{k_j}$. For all j we have $|x_{k_j} - y_{k_j}| \leq \frac{1}{k_j}$ hence $y_{k_j} - \frac{1}{k_j} \leq x_{k_j} \leq y_{k_j} + \frac{1}{k_j}$. Since $y_{k_j} \to c$ and $\frac{1}{k_j} \to 0$ we have $y_{k_j} \pm \frac{1}{k_j} \to c$ and hence $x_{k_j} \to c$ by the Squeeze Theorem. Since f is continuous at c and $x_{k_j} \to c$ and $y_{k_j} \to c$, we have $f(x_{k_j}) \to f(c)$ and $f(y_{k_j}) \to f(c)$ by the Sequential Characterization of Continuity. Since $f(x_{k_j}) \to c$ and $f(y_{k_j}) \to c$ we have $f(x_{k_j}) - f(y_{k_j}) \to 0$. But this implies that we can choose j so that $|f(x_{k_j}) - f(y_{k_j})| \leq \epsilon$, giving the desired contradiction.