4. The Generalized Cross Product

4.1 Definition: Given vectors $u_1, u_2, \cdots, u_k \in \mathbb{R}^n$, we define the **parallelotope** on $u_1, \cdots, u_k$ to be the set

$$P(u_1, \cdots, u_k) = \left\{ \sum_{j=1}^{k} t_i u_i \middle| 0 \leq t_i \leq 1 \text{ for all } i \right\}.$$

We define the **volume** of this parallelotope, denoted by $V(u_1, \cdots, u_k)$, recursively by $V(u_1) = \|u_1\|$ and

$$V(u_1, \cdots, u_k) = V(u_1, \cdots, u_{k-1}) \big\| \text{Proj}_{U^\perp}(u_k) \big\|$$

where $U = \text{Span}\{u_1, \cdots, u_{k-1}\}$.

4.2 Theorem: L*et* $u_1, \cdots, u_k \in \mathbb{R}^n$ *and let* $A = (u_1, \cdots, u_k) \in M_{n \times k}(\mathbb{R})$. *Then*

$$V(u_1, \cdots, u_n) = \sqrt{\det(A^T A)}.$$

Proof: We prove the theorem by induction on k. Note that when $k = 1$, $u_1 \in \mathbb{R}^n$ and $A = u_1 \in M_{n \times 1}(\mathbb{R})$, we have $V(u_1) = \|u_1\| = \sqrt{u_1 \cdot u_1} = \sqrt{{u_1}^T u_1} = \sqrt{A^T A}$, as required. Let $k \geq 2$ and suppose, inductively, that when $A = (u_1, \cdots, u_{k-1}) \in M_{n \times k-1}$ we have $\det(A^T A) > 0$ and $V(u_1, \cdots, u_{k-1}) = \sqrt{\det(A^T A)}$. Let $B = (u_1, \cdots, u_k) = (A, u_k)$. Let $U = \text{Span}\{u_1, \cdots, u_{k-1}\} = \text{Col}(A)$. Let $v = \text{Proj}_U(u_k)$ and $w = \text{Proj}_{U^\perp}(u_k)$. Note that $v \in U = \text{Col}(A)$ and $w \in U^\perp = \text{Null}(A^T)$. Then we have $u_k = v + w$ so that $B = (A, v + w)$. Since $v \in \text{Col}(A)$, the matrix B can be obtained from the matrix (A, w) by performing elementary column operations of the type $C_k \mapsto C_k + tC_i$. Let E be the product of the elementary matrices corresponding to these column operations, and note that $B = (A, v + w) = (A, w)E$. Since the row operations $C_k \mapsto C_k + tC_i$ do not alter the determinant, E is a product of elementary matrices of determinant 1, so we have $\det(E) = 1$. Since $\det(E) = 1$ and $w \in \text{Null}(A^T)$ we have

$$\begin{aligned} \det(B^T B) &= \det\Big(E^T (A, w)^T (A, w) E\Big) = \det\left(\begin{pmatrix} A^T \\ w^T \end{pmatrix} \begin{pmatrix} A & w \end{pmatrix} \right) \\ &= \det \begin{pmatrix} A^T A & A^T w \\ w^T A & w^T w \end{pmatrix} = \begin{pmatrix} A^T A & 0 \\ 0 & \|w\|^2 \end{pmatrix} = \det(A^T A)\, \|w\|^2. \end{aligned}$$

By the induction hypothesis, we can take the square root on both sides to get

$$\sqrt{\det(B^T B)} = \sqrt{\det(A^T A)}\, \|w\| = V(u_1, \cdots, u_{k-1})\ \|w\| = V(u_1, \cdots, u_k).$$

4.3 Note: In the special case that $A = (u_1, u_2, \cdots, u_n) \in M_n(\mathbb{R})$, we have

$$V(u_1, \cdots, u_n) = \sqrt{\det(A^T A)} = \sqrt{\det(A)^2} = \big|\det(A)\big|.$$

4.4 Remark: There is a similar formula for the volume of an l-simplex in $\mathbb{R}^n$. For the l-simplex $S = [a_0, a_1, \cdots, a_l]$, if we let $A = (u_1, u_2, \cdots, u_l) \in M_{n \times l}(\mathbb{R})$ where $u_k = a_k - a_0$, then the volume of S is given by

$$V[a_0, a_1, \cdots, a_l] = \frac{1}{l\,!} V(u_1, \cdots, u_l) = \frac{1}{l\,!} \sqrt{\det(A^T A)}.$$

4.5 Definition: Let F be a field. For $n \geq 2$ we define the **cross product**

$$X : M_{n\times(n-1)}(F) = \prod_{k=1}^{n-1} F^n \to F^n$$

as follows. Given $A = (u_1, u_2, \cdots, u_{n-1}) \in M_{n\times(n-1)}(F)$, we define $X(A)$, also written as $X(u_1, u_2, \cdots, u_{n-1})$, to be the vector in F^n with entries

$$X(A)_j = X(u_1, u_2, \cdots, u_{n-1})_j = (-1)^{n+j} \det A^{(j)}$$

where $A^{(j)} \in M_{n-1}(F)$ is the matrix obtained from A by removing the j^{th} row. For $u \in F^2$ we write $X(u)$ as $u^\times$, and for $u, v \in F^3$ we write $X(u, v)$ as $u \times v$.

4.6 Example: Given $u \in F^2$ we have

$$u^\times = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^\times = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}.$$

Given $u, v \in F^3$ we have

$$u \times v = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \\ -\det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix} \\ \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

4.7 Note: Because the determinant is n-linear and alternating, it follows that the cross product is $(n-1)$-linear and alternating. Thus for $u_i, v, w \in F^n$ and $t \in F$ we have

(1) $X(u_1, \cdots, v + w, \cdots, u_{n-1}) = X(u_1, \cdots, v, \cdots, u_{n-1}) + X(u_1, \cdots, w, \cdots, u_{n-1})$,
(2) $X(u_1, \cdots, t\, u_k, \cdots, u_{n-1}) = t\, X(u_1, \cdots, u_k, \cdots, u_{n-1})$,
(3) $X(u_1, \cdots, u_k, \cdots, u_l, \cdots, u_{n-1}) = -X(u_1, \cdots, u_l, \cdots, u_k, \cdots, u_{n-1})$.

4.8 Definition: Recall that for $u_1, \cdots, u_n \in \mathbb{R}^n$, the set $\{u_1, \cdots, u_n\}$ is a basis for $\mathbb{R}^n$ if and only if $\det(u_1, \cdots, u_n) \neq 0$. For an ordered basis $\mathcal{A} = (u_1, \cdots, u_n)$, we say that $\mathcal{A}$ is **positively oriented** when $\det(u_1, \cdots, u_n) > 0$ and we say that $\mathcal{A}$ is **negatively oriented** when $\det(u_1, \cdots, u_n) < 0$.

4.9 Theorem: *(Properties of the Cross Product) For* $u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}, w \in \mathbb{R}^n$,

(1) $X(u_1, \cdots, u_{n-1}) \bullet w = \det(u_1, \cdots, u_{n-1}, w)$,
(2) $X(u_1, \cdots, u_{n-1}) \bullet u_k = 0$ *for* $1 \leq k < n$.
(3) $X(u_1, \cdots, u_{n-1}) = 0$ *if and only if* $\{u_1, \cdots, u_{n-1}\}$ *is linearly dependent.*
(4) When $w = X(u_1, \cdots, u_{n-1}) \neq 0$ *we have* $\det(u_1, \cdots, u_{n-1}, w) > 0$ *so that the* n*-tuple* $(u_1, \cdots, u_{n-1}, w)$ *is a positively oriented basis for* $\mathbb{R}^n$,
(5) $\big\|X(u_1, \cdots, u_{n-1})\big\|$ *is equal to the volume of the parallelotope on* $u_1, \cdots, u_{n-1}$,
(6) $X(u_1, \cdots, u_{n-1}) \bullet X(v_1, \cdots, v_{n-1}) = \det(B^T A)$ *where* $A = (u_1, \cdots, u_{n-1}) \in M_{n\times n-1}(\mathbb{R})$ *and* $B = (v_1, \cdots, v_{n-1}) \in M_{n\times n-1}(\mathbb{R})$, *and*
(7) $X\big(u_1, \cdots, u_{n-2}, X(v_1, \cdots, v_{n-1})\big) = \sum_{i=1}^{n-1} (-1)^{n+i} \det\big((B^T A)^{(i)}\big)\, v_i$ *where* $A = (u_1, \cdots, u_{n-2})$ *and* $B = (v_1, \cdots, v_{n-1})$, *and* $(B^T A)^{(i)}$ *is obtained from* $B^T A$ *by removing the* i^{th} *row.*

Proof: Since $X(u_1,\cdots,u_{n-1})=\sum\limits_{i=1}^{n}(-1)^{n+i}\det A^{(i)}e_i$ we have

$$X(u_1,u_2,\cdots,u_{n-1})\cdot w=\sum_{i=1}^{n}(-1)^{n+i}\det A^{(i)}w_i=\det(u_1,\cdots,u_{n-1},w),$$

where the last equality follows by expanding the determinant along the last column. This proves Part (1), and Part (2) follows from Part (1) since $\det(u_1,\cdots,u_k,\cdots,u_{n-1},u_k)=0$.

To prove Part (3), let $A=(u_1,\cdots,u_{n-1})$. Then $\{u_1,\cdots,u_{n-1}\}$ is linearly independent if and only $\operatorname{rank}(A)=n-1$ if and only if some set of $n-1$ rows of A are linearly independent if and only if $A^{(i)}$ is invertible for some index i if and only if $X(u_1,\cdots,u_{n-1})\neq 0$.

Part (4) holds because when $w=X(u_1,\cdots,u_{n-1})\neq 0$ we have $\|w\|^2>0$ so that

$$0<\|w\|^2=w\cdot w=X(u_1,\cdots,u_{n-1})\cdot w=\det(u_1,\cdots,u_{n-1},w).$$

To prove Part (6), let $x=X(u_1,\cdots,u_{n-1})$, $y=X(v_1,\cdots,v_{n-1})$, $A=(u_1,\cdots,u_{n-1})$ and $B=(v_1,\cdots,v_{n-1})$. Using Part (1) we see that $x\cdot y=\det(u_1,\cdots,u_{n-1},y)=\det(A,y)$ and also $x\cdot y=\det(v_1,\cdots,v_{n-1},x)=\det(B,x)$, and so

$$(x\cdot y)^2=\det(A,y)\det(B,x)=\det\big((B,x)^T(A,y)\big)=\det\begin{pmatrix}B^TA & B^Ty\\ x^TA & x^Ty\end{pmatrix}.$$

By Part (2), x is perpendicular to the columns of A and y is perpendicular to the columns of B and so we have $A^Tx=0=B^Ty$ and so

$$(x\cdot y)^2=\det\begin{pmatrix}B^TA & 0\\ 0 & x\cdot y\end{pmatrix}=(x\cdot y)\det(B^TA).$$

When $x\cdot y\neq 0$, we can divide both sides by $x\cdot y$ to get $x\cdot y=\det(B^TA)$, as required.

We shall now provide two proofs to deal with the case in which $x\cdot y=0$. For the first proof, we consider both sides of the above equality, namely $(x\cdot y)^2$ and $(x\cdot y)\det(B^TA)$, to be polynomials in the entries of the vectors u_i and v_j. By unique factorization of polynomials (in many variables), we obtain $(x\cdot y)=\det(B^TA)$, as required.

Here is an alternate proof. Suppose that $x\cdot y=0$. First we consider the case that $x=0$ or $y=0$. In this case, either $\operatorname{rank}(A)<n-1$ or $\operatorname{rank}(B)<n-1$, and in either case we have $\operatorname{rank}(B^TA)<n-1$ so that B^TA is not invertible, hence $\det(B^TA)=0=x\cdot y$. Finally, we consider the case that $x\cdot y=0$ with $x\neq 0$ and $y\neq 0$. In this case, since $x\cdot y=0$ we have $y\in\operatorname{Span}\{x\}^\perp$. Since $x\neq 0$, the set $\{u_1,\cdots,u_{n-1}\}$ is linearly independent by Part (3) and so we have $y\in\operatorname{Span}\{x\}^\perp=\operatorname{Span}\{u_1,\cdots,u_{n-1}\}=\operatorname{Col}(A)$. But also, by Part (2), we have $y\in\operatorname{Span}\{v_1,\cdots,v_{n-1}\}^\perp=\operatorname{Col}(B)^\perp=\operatorname{Null}(B^T)$. Since $0\neq y\in\operatorname{Col}(A)$ we can write $y=At$ for some $0\neq t\in\mathbb{R}^{n-1}$, and since $y\in\operatorname{Null}(B^T)$ we have $0=B^Ty=B^TAt$. Since $t\neq 0$ and $B^TAt=0$ it follows that B^TA is not invertble, so again we find that $\det(B^TA)=0=x\cdot y$. This completes the proof of Part (6).

Note that Part (5) follows from Part (6). Indeed when $A=(u_1,\cdots,u_{n-1})$ we have

$$\big\|X(u_1,\cdots,u_{n-1})\big\|^2=X(u_1,\cdots,u_{n-1})\cdot X(u_1,\cdots,u_{n-1})=\det(A^TA)$$

and so

$$\big\|X(u_1,\cdots,u_{n-1})\big\|=\sqrt{\det(A^TA)}=V(u_1,\cdots,u_{n-1}).$$

In order to prove Part (7), we shall obtain a change of variables formula for the cross product. Let $A = (u_1, \cdots, u_{n-1}) \in M_{n\times(n-1)}(\mathbb{R})$ and let $P = (v_1, \cdots, v_n) \in M_n(\mathbb{R})$. Note that the i^{th} entry of $P^T X(PA)$ is

$$\big(P^T X(PA)\big)_i = {v_i}^T X(PA) = X(PA) \bullet v_i = \det(PA, v_i).$$

Recall that $\operatorname{Cof}(P)\, P = P \operatorname{Cof}(P) = \det(P)\, I$, where $\operatorname{Cof}(P)$ is the cofactor matrix of P, so we have

$$\begin{aligned}(\det P)^n \big(P^T X(PA)\big)_i &= \det\big(P \operatorname{Cof}(P)\big) \det(PA, v_i) = \det(P) \det\big(\operatorname{Cof}(P)PA, \operatorname{Cof}(P)v_i\big)\\ &= \det(P) \det\big((\det P)A, (\operatorname{Cof}(P)P)_i)\big) = \det(P) \det\big((\det P)A, (\det P)e_i\big)\\ &= (\det P)^{n+1} \det(A, e_i) = (\det P)^{n+1}(-1)^{n+i} \det A^{(i)} = (\det P)^{n+1} X(A)_i.\end{aligned}$$

Thus $(\det P)^n P^T X(PA) = (\det P)^{n+1} X(A)$. When P is invertible, we can divide both sides by $(\det P)^n$ to get $P^T X(PA) = (\det P)\, X(A)$. Even when P is not invertible, we can regard both sides of the equality $(\det P)^n P^T X(PA) = (\det P)^{n+1} X(A)$ as polynomials in the entries of the vectors u_i and v_j, and then by unique factorization we obtain the change of variables formula

$$P^T X(PA) = (\det P)\, X(A).$$

Alternatively, replacing P by P^T, we obtain

$$P\, X(P^T A) = (\det P)\, X(A).$$

Finally, let us prove Part (7). Let $A = (u_1, \cdots, u_{n-2})$ and $B = (v_1, \cdots, v_{n-1})$, and let $y = X(B) = X(v_1, \cdots, v_{n-1})$, so that we have

$$X\big(u_1, \cdots, u_{n-2}, X(v_1, \cdots, v_{n-1})\big) = X(A, y).$$

Let $P = (B, y) = (v_1, \cdots, v_{n-1}, y)$. Note that

$$\det P = X(v_1, \cdots, v_{n-1}) \bullet y = y \bullet y = \|y\|^2.$$

By the above change of variables formula, we have

$$\begin{aligned}\|y\|^2 X(A, y) &= (\det P) X(A, y) = P\, X\big(P^T(A, y)\big)\\ &= P\, X\left(\begin{pmatrix} B^T \\ y^T \end{pmatrix} (A, y)\right) = P\, X \begin{pmatrix} B^T A & B^T y \\ y^T A & y^T y \end{pmatrix} = P\, X \begin{pmatrix} B^T A & 0 \\ y^T A & \|y\|^2 \end{pmatrix}\\ &= P\left(\left(\sum_{i=1}^{n-1} (-1)^{n+i} \det \begin{pmatrix} (B^T A)^{(i)} & 0 \\ y^T A & \|y\|^2 \end{pmatrix} e_i\right) + 0 \cdot e_n\right)\\ &= (v_1, \cdots, v_{n-1}, y) \left(\sum_{i=1}^{n-1} (-1)^{n+i} \|y\|^2 \det(B^T A)^{(i)}\ e_i\ + 0 \cdot e_n\right)\\ &= \|y\|^2\ \sum_{i=1}^{n-1} (-1)^{n+i} \det\big((B^T A)^{(i)}\big) v_i\end{aligned}$$

Regarding both sides of the equality $\|y\|^2 X(A, y) = \|y\|^2 \sum_{i=1}^{n-1} (-1)^{n+i} \det\big((B^T A)^{(i)}\, v_i$ as polynomials in the entries of the vectors u_i and v_j, we can divide both sides by $\|y\|^2$ to obtain $X(A, y) = \sum_{i=1}^{n-1} \det\big((B^T A)^{(i)}\, v_i$, as required.