

AMATH/PMATH 331 Real Analysis, Problems for Chapter 6

- 1:** (a) Let  $A = \left\{x \in \mathbf{R}^2 \mid \|x\| = \frac{n+1}{n} \text{ for some } n \in \mathbf{Z}^+\right\}$ . Determine whether  $A$  is complete or compact.  
 (b) Let  $A$  be the set of points  $(a, b, c, d) \in \mathbf{R}^4$  such that the points  $(0, 0)$ ,  $(a, b)$  and  $(c, d)$  are the vertices of a right-angled triangle in  $\mathbf{R}^2$  whose area is equal to 1. Determine whether  $A$  is complete or compact.
- 2:** (a) Let  $M_2(\mathbf{R})$  be the set of  $2 \times 2$  matrices with entries in  $\mathbf{R}$  and let  $O_2(\mathbf{R}) = \{A \in M_2(\mathbf{R}) \mid A^T A = I\}$ . Define  $F : \mathbf{R}^4 \rightarrow M_2(\mathbf{R})$  by  $F(x, y, z, w) = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$  and let  $A = \left\{(x, y, z, w) \in \mathbf{R}^4 \mid F(A) \in O_2(\mathbf{R})\right\}$ . Show that  $A$  is compact.  
 (b) Recall from linear algebra (or verify) that the space  $M_{n \times m}(\mathbf{R})$  of  $n \times m$  matrices with entries in  $\mathbf{R}$  is an inner-product space with inner product given by  $\langle A, B \rangle = \text{trace}(B^T A) = \sum_{k=1}^n \sum_{\ell=1}^m A_{k,\ell} B_{k,\ell}$ , and with standard orthonormal basis  $\{E_{k,\ell} \mid 1 \leq k \leq n, 1 \leq \ell \leq m\}$  where  $E_{k,\ell}$  is the  $n \times m$  matrix whose  $(k, \ell)$  entry is equal to 1 and all other entries are zero, and the linear map  $L = L_{n,m} : M_{n \times m}(\mathbf{R}) \rightarrow \mathbf{R}^{nm}$  given by  $L(E_{k,\ell}) = e_{(k-1)n+\ell}$  is an inner product space isomorphism. Show that the set  $S = \{A \in M_{n \times m}(\mathbf{R}) \mid A^T A = I\}$  is compact.
- 3:** (a) Show that  $(\ell_\infty, d_\infty)$  is complete.  
 (b) Show that  $(\ell_2, d_2)$  is complete.
- 4:** For each of the following sets  $A$ , determine whether  $A$  is complete and whether  $A$  is compact.  
 (a)  $A = \overline{B}_2(0, 1) = \{a = (a_n)_{n \geq 1} \in \ell_2 \mid \|a\|_2 \leq 1\} \subseteq \ell_2$ , using the metric  $d_2$ .  
 (b)  $A = \{a = (a_n)_{n \geq 1} \in \ell_1 \mid |a_n| \leq \frac{1}{n} \text{ for all } n \in \mathbf{Z}^+\} \subseteq \ell_1$ , using the metric  $d_1$ .  
 (c)  $A = \{a = (a_n)_{n \geq 1} \in \ell_2 \mid |a_n| \leq \frac{1}{n+1} \text{ for all } n \in \mathbf{Z}^+\} \subseteq \ell_2$ , using the metric  $d_2$ .
- 5:** (a) Show that the closed unit ball  $\overline{B}_\infty(0, 1)$  is not compact in  $\mathcal{C}[0, 1]$ , using the metric  $d_\infty$ .  
 (b) Show that  $\mathcal{C}[-1, 1]$  is not complete using the metric  $d_1$ .
- 6:** (a) Let  $X$  be a metric space, let  $A \subseteq X$  be compact, and let  $S$  be an open cover for  $A$  in  $X$ . Show that there exists  $r > 0$  with the property that for every  $a \in A$  there exists  $U \in S$  such that  $B(a, r) \subseteq U$ .  
 (b) Let  $X$  be a compact metric space. Let  $(f_n)$  be a sequence in  $\mathcal{C}(X)$  which converges pointwise to a function  $f \in \mathcal{C}(X)$ . Show that if  $(f_n)$  is *increasing*, meaning that  $\forall n \in \mathbf{N} \forall x \in X f_n(x) \leq f_{n+1}(x)$ , then the convergence is uniform.  
 (c) Show that the requirements in Part (b) that  $X$  is compact and that  $(f_n)$  is increasing are both necessary.
- 7:** (Absolute convergence implies convergence) Let  $X$  be a normed linear space. For a sequence  $(x_k)_{k \geq 1}$  in  $X$ , the  $n^{\text{th}}$  *partial sum* of  $(x_k)_{k \geq 1}$  is the element  $s_n = \sum_{k=1}^n x_k \in X$ , the *series*  $\sum_{k=1}^\infty x_k$  is, by definition, equal to the sequence of partial sums  $(s_n)_{n \geq 1}$ , we say the series  $\sum_{k=1}^\infty x_k$  *converges* in  $X$  when the sequence of partial sums  $(s_n)_{n \geq 1}$  converges in  $X$  and then the *sum* of the series (also denoted by  $\sum_{k=1}^\infty x_k$ ) is defined to be the limit of the sequence of partial sums in  $X$ . Show that  $X$  is complete if and only if  $X$  has the property that for every sequence  $(x_k)_{k \geq 1}$  in  $X$ , if  $\sum_{k=1}^\infty \|x_k\|$  converges in  $\mathbf{R}$  then  $\sum_{k=1}^\infty x_k$  converges in  $X$ .
- 8:** Let  $X$  be a metric space.  
 (a) Show that  $X$  is complete if and only if every decreasing sequence of closed balls
- $$\overline{B}(a_1, r_1) \supset \overline{B}(a_2, r_2) \supset \overline{B}(a_3, r_3) \supset \dots$$
- in  $X$  with  $r_n \rightarrow 0$  has a non-empty intersection.  
 (b) Show that the requirement in Part (a) that  $r_n \rightarrow 0$  is necessary.