Chapter 4. Separability and Completeness

Separability

4.1 Note: Let X be a metric space. Recall that for $A \subseteq X$ we say that A is **dense** in X when $\overline{A} = X$. Also recall that $\overline{A} = A \cup A'$ where A' is the set of limit points of A and so, by the definition of limit points, it follows that A is dense in X if and only if every open ball in X contains a point in A. By the sequential characterization of the closure, we can say that A is dense in X if and only if for every $a \in X$ there exists a sequence (x_n) in A with $x_n \to a$ in X.

4.2 Definition: Let X be a metric space (or a topological space). We say that X is **separable** when it has a finite or countable dense subset.

4.3 Definition: Let X be a topological space. A **basis** (or a **base**) for the topology on X is a set S of open sets in X with the property that for every subset $U \subseteq X$, U is open if and only if for every point $a \in U$ there exists a basic set $B \in S$ with $a \in B \subseteq U$.

4.4 Example: In a metric space X, the set of open balls $S = \{B(a,r) \big| a \in X, 0 < r \in \mathbb{R}\}$ is a basis for the metric topology on X.

4.5 Theorem: *Let X be a metric space.*

(1) If X is separable then there is a finite or countable basis for the metric topology on X.
(2) If every infinite subset of X has a limit point then X is separable.
(3) If X is separable then every subspace of X is separable.

Proof: We prove Parts 1 and 3 and leave the proof of Part 2 as an exercise. To prove Part 1, let X be separable and choose a finite or countable dense subset $P = \{p_1, p_2, p_3, \cdots\} \subseteq X$. Let $S = \{B(p_k, \frac{1}{\ell}) \,\big|\, k, \ell \in \mathbb{Z}^+\}$. Note that S is finite or countable because the map $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \to S$ given by $F(k,\ell) = B(p_k, \frac{1}{\ell})$ is surjective (F might not be injective even if the elements p_k are distinct). We claim that S is a basis for the topology on X.

Let $U \subseteq X$. Suppose first that that U is open in X. Let $a \in U$. Since U is open in X we can choose $r > 0$ such that $B(a, 2r) \subseteq U$. Choose $\ell \in \mathbb{Z}^+$ so that $\frac{1}{\ell} < r$. Since $P = \{p_1, p_2, \cdots\}$ is dense in X we can choose an index $k \in \mathbb{Z}^+$ so that $p_k \in B(a, \frac{1}{\ell})$. Since $d(p_k, a) < \frac{1}{\ell}$, we have $a \in B(p_k, \frac{1}{\ell})$. Also note that for all $x \in B(p_k, \frac{1}{\ell})$, we have $d(x,a) \leq d(x, p_k) + d(p_k, a) < \frac{1}{\ell} + \frac{1}{\ell} < 2r$ so that $a \in B(p_k, \frac{1}{\ell}) \subseteq B(a, 2r) \subseteq U$.

Now suppose that U has the property that for every $a \in U$ there exists $B \in S$ such that $a \in B \subseteq U$. For each $a \in U$ choose $B_a \in S$ such that $a \in B_a \subseteq U$. Then $U = \bigcup_{a \in U} B_a$, which is open (it a union of open sets). Thus S is a basis for the topology on X.

To prove Part 3, let X be separable and let $\emptyset \neq Y \subseteq X$. Since X is separable we can choose a finite or countable dense subset $P \subseteq X$, say $P = \{p_1, p_2, p_3, \cdots\}$. Recall from Part 1 that the set $S = \{B(p_k, \frac{1}{\ell}) \,\big|\, k, \ell \in \mathbb{Z}^+\}$ is a finite or countable basis for the topology on X. Fix an element $b \in Y$. For each $k, \ell \in \mathbb{Z}^+$, if $B(p_k, \frac{1}{\ell}) \cap Y \neq \emptyset$ then then choose an element $q_{k,\ell} \in B(p_k, \frac{1}{\ell}) \cap Y$ and if $B(p_k, \frac{1}{\ell}) \cap Y = \emptyset$ then choose $q_{k,\ell} = b$. Let $Q = \{q_{k,\ell} \big| k, \ell \in \mathbb{Z}^+\}$. Note that Q is a finite or countable subset of Y. We claim that Q is dense in Y. Let $y \in Y$ and let $\epsilon > 0$. Since S is a basis for the topology on X we can choose $k, \ell \in \mathbb{Z}^+$ such that $y \in B(p_k, \frac{1}{\ell}) \subseteq B(y, \epsilon)$. Since $y \in B(p_k, \frac{1}{\ell}) \cap Y$ so that $B(p_k, \frac{1}{\ell}) \cap Y \neq \emptyset$, we have $q_{k,\ell} \in B(p_k, \frac{1}{\ell}) \cap Y$. Since $q_{k,\ell} \in B(p_k, \frac{1}{\ell}) \subseteq B(y, \epsilon)$, we have $d(q_{k,\ell}, y) < \epsilon$. Thus the set Q is dense in Y, as claimed.

4.6 Example: Euclidean space $(\mathbb{R}^n, d_2)$ is separable with $\mathbb{Q}^n$ as a countable dense subset. Similarly, the complex space $(\mathbb{C}^n, d_2)$ is separable with $\mathbb{Q}[i]^n$ as a countable dense subset, where $\mathbb{Q}[i] = \{x+iy \,|\, x, y \in \mathbb{Q}\}$.

4.7 Theorem: *The spaces (ℓ_1, d_1) and (ℓ_2, d_2) are separable, and (ℓ_∞, d_∞) is not.*

Proof: We give the proof that $(\ell_1(\mathbb{R}), d_1)$ is separable, and that (ℓ_∞, d_∞) is not. We leave the cases $(\ell_1(\mathbb{C}), d_1)$ and (ℓ_2, d_2) as an exercise. First we claim that (ℓ_1, d_1) is separable when $\mathbb{F} = \mathbb{R}$. For each $n \in \mathbb{Z}^+$, let

$$A_n = \left\{ a = (a_k)_{k\geq 1} \in \mathbb{R}^\infty \middle| a_k \in \mathbb{Q} \text{ for } k \leq n \text{ and } a_k = 0 \text{ for } k > n \right\}$$

and let $\mathbb{Q}^\infty = \bigcup\limits_{n=1}^{\infty} A_n$. Note that for each n there is a natural bijection $F : \mathbb{Q}^n \to A_n$ given by $F(a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_n, 0, 0, \cdots)$, so each A_n is countable, and hence $\mathbb{Q}^\infty$ is countable (by Theorem 1.20). We claim that $\mathbb{Q}^\infty$ is dense in (ℓ_1, d_1). Let $b = (b_k)_{k\geq 1} \in \ell_1$. Let $\epsilon > 0$. Choose $n \in \mathbb{Z}^+$ so that $\sum\limits_{k=n+1}^{\infty} |b_k| < \frac{\epsilon}{2}$. For each $k \leq n$ choose $a_k \in \mathbb{Q}$ so that $|a_k - b_k| < \frac{\epsilon}{2n}$, and for each $k > n$ let $a_k = 0$. Then $a = (a_k) \in A_n \subseteq \mathbb{Q}^\infty$ and we have

$$\|a - b\|_1 = \sum_{k=1}^{\infty} |a_k - b_k| = \sum_{k=1}^{n} |a_k - b_k| + \sum_{k=n+1}^{\infty} |b_k| < n \cdot \tfrac{\epsilon}{2n} + \tfrac{\epsilon}{2} = \epsilon\,.$$

Thus $\mathbb{Q}^\infty$ is a countable dense subset of (ℓ_1, d_1), so (ℓ_1, d_1) is separable.

Next we claim that (ℓ_∞, d_∞) is not separable (when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). For each $A \subseteq \mathbb{Z}^+$, let $e_A = (e_{A,k})_{k\geq 1}$ where

$$e_{A,k} = \begin{cases} 1 \text{ , if } k \in A \\ 0 \text{ , if } k \notin A\,. \end{cases}$$

Note that for $A \neq B \subseteq \mathbb{Z}^+$ we have $\|e_A - e_B\|_\infty = 1$, so the balls $B_\infty(e_A, \frac{1}{2})$ are disjoint. Let $P \subseteq \ell_\infty$ be any dense subset. For each $A \subseteq \mathbb{Z}^+$, choose $p_A \in P \cap B_\infty(e_A, \frac{1}{2})$. Since the balls are disjoint, the map $F : \mathcal{P}(\mathbb{Z}^+) \to P$ given by $F(A) = p_A$ is injective, so we have

$$2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| \leq |P|\,.$$

Thus (ℓ_∞, d_∞) is not separable.

4.8 Example: As an exercise, show that the space $(\mathcal{B}[a, b], d_\infty)$ of bounded functions on the interval $[a, b]$ is not separable (consider characteristic functions χ_A for appropriate sets $A \subseteq [a, b]$).

4.9 Remark: Later (in Chapter 6) we will show that the space $(\mathcal{C}[a, b], d_\infty)$ of continuous real valued functions on the interval $[a, b]$ is separable. Once we have proven this, it will follow that every subspace of $\mathcal{C}[a, b]$ is separable, using the supremum metric.

Completeness

4.10 Definition: A sequence $(x_n)_{n\geq 1}$ in a metric space X is called a **Cauchy sequence** when it has the property that for all $\epsilon > 0$ there exists an index $m \in \mathbb{Z}^+$ such that for all indices $k, \ell \geq m$ we have $d(x_k, x_\ell) < \epsilon$.

4.11 Theorem: *Let X be a metric space.*

(1) Every Cauchy sequence in X is bounded.
(2) Every convergent sequence in X is Cauchy.
(3) If some subsequence of a Cauchy sequence (x_n) converges, then (x_n) converges.

Proof: To prove Part 1, let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Choose $m \in \mathbb{Z}^+$ such that $k, \ell \geq m \Longrightarrow d(x_k, x_\ell) \leq 1$ and note that, in particular, we have $d(x_k, x_m) \leq 1$ for all $k \geq m$. Let $a = x_m$ and choose $r > \max\big\{d(x_1, a), d(x_2, a), \cdots, d(x_{m-1}, a), 1\big\}$. Then for all $n \in \mathbb{Z}^+$ we have $d(x_n, a) < r$ so the sequence (x_n) is bounded, as required.

To prove Part 2, let $(x_n)_{n\geq 1}$ be a convergent sequence in X and let $a = \lim\limits_{n\to\infty} x_n$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that $n \geq m \Longrightarrow d(x_n, a) < \frac{\epsilon}{2}$. Then for all $k, \ell \geq m$ we have

$$d(x_k, x_\ell) \leq d(x_k, a) + d(a, x_\ell) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon,$$

so the sequence (x_n) is Cauchy, as required.

To prove Part 3, let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X, let $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$, suppose that $(x_{n_k})_{k\geq 1}$ converges, and let $a = \lim\limits_{k\to\infty} x_{n_k}$. Let $\epsilon > 0$. Since (x_n) is Cauchy we can choose $m \in \mathbb{Z}^+$ so that $k, \ell \geq m \Longrightarrow d(x_k, x_\ell) < \frac{\epsilon}{2}$. Since $\lim\limits_{k\to\infty} n_k = \infty$ and $\lim\limits_{k\to\infty} x_{n_k} = a$, we can choose an index ℓ such that $n_\ell \geq m$ and $d(x_{n_\ell}, a) < \frac{\epsilon}{2}$. Then for all $k \geq m$ we have

$$d(x_k, a) \leq d(x_k, x_{n_\ell}) + d(x_{n_\ell}, a) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

4.12 Definition: A metric space X is called **complete** when every Cauchy sequence in X converges in X. A complete inner product space is called a **Hilbert space**, and a complete normed linear space is called a **Banach space**.

4.13 Theorem: *Let X be a complete metric space and let $A \subseteq X$. Then A is complete if and only if A is closed in X*

Proof: Suppose that A is closed in X. Let (x_n) be a Cauchy sequence in A. Since X is complete, (x_n) converges in X. Since A is closed in X and (x_n) is a sequence in A which converges in X, we have $\lim\limits_{n\to\infty} x_n \in A$ by Theorem 3.17 (The Sequential Characterization of Closed Sets). Thus every Cauchy sequence in A converges in A, so A is complete.

Suppose, conversely, that A is complete. Let $a \in A'$, that is let $a \in X$ be a limit point of A. Since $a \in A'$, by Theorem 3.5 (The Sequential Characterization of Limit Points) we can choose a sequence (x_n) in A (indeed in $A \setminus \{a\}$) with $\lim\limits_{n\to\infty} x_n = a$. Since (x_n) converges in X, it is Cauchy. Since (x_n) is Cauchy and A is complete, (x_n) converges in A, that is $a = \lim\limits_{n\to\infty} x_n \in A$.

4.14 Example: Recall, from MATH 247 or PMATH 333, that $\big(\mathbb{R}^n, d_2\big)$ is complete. Note that $(\mathbb{C}^n, d_2)$ is also complete because $(\mathbb{C}^n, d_2) = (\mathbb{R}^{2n}, d_2)$. It follows that every closed subset $A \subseteq \mathbb{R}^n$ (or $A \subseteq \mathbb{C}^n$) is complete (using the standard metric d_2).

4.15 Example: Note that completeness is not invariant under homeomorphism. For example, $\mathbb{R}$ is homeomorphic to $(0,1) \subseteq \mathbb{R}$, but $\mathbb{R}$ is complete while $(0,1)$ is not.

4.16 Theorem: *Every finite-dimensional normed linear space is complete.*

Proof: Let U be an n-dimensional normed linear space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $\{u_1,\cdots,u_n\}$ be a basis for the vector space U and let $F:\mathbb{F}^n\to U$ be the associated vector space isomorphism given by $F(t)=\sum\limits_{k=1}^n t_ku_k$. Recall, from Theorem 3.38, that both F and F^{-1} are Lipschitz continuous. Let L be a Lipshitz constant for F and let M be a Lipschitz constant for F^{-1}. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in U. For each $n\in\mathbb{Z}^+$, let $t_n=F^{-1}(x_n)\in\mathbb{F}^n$. Note that (t_n) is a Cauchy sequence in $\mathbb{F}^n$ because

$$\|t_k-t_\ell\|_2=\left\|F^{-1}(x_k)-F^{-1}(x_\ell)\right\|_2\leq M\|x_k-x_\ell\|.$$

Since (t_n) is a Cauchy sequence in $\mathbb{F}^n$ and $\mathbb{F}^n$ is complete, (t_n) converges in $\mathbb{F}^n$. Let $s=\lim\limits_{n\to\infty}t_n\in\mathbb{F}^n$ and let $a=F(s)\in U$. Then we have $\lim\limits_{n\to\infty}x_n=a$ because

$$\|x_n-a\|=\left\|F(t_n)-F(s)\right\|\leq\ L\|t_n-s\|_2.$$

4.17 Corollary: *The metric spaces $(\mathbb{F}^n,d_1)$, $(\mathbb{F}^n,d_2)$ and $(\mathbb{F}^n,d_\infty)$ are all complete.*

4.18 Theorem: *The metric spaces (ℓ_1,d_1), (ℓ_2,d_2) and (ℓ_∞,d_∞) are all complete.*

Proof: We prove that (ℓ_1,d_1) is complete and we leave the proof that (ℓ_2,d_2) and (ℓ_∞,d_∞) are complete as an exercise. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in ℓ_1. For each $n\in\mathbb{Z}^+$, write $a_n=(a_{n,k})_{k\geq 1}=(a_{n,1},a_{n,2},a_{n,3},\cdots)$. Since $a_n\in\ell_1$ we have $\sum\limits_{k=1}^\infty|a_{n,k}|<\infty$. Since $(a_n)_{n\geq 1}$ is Cauchy, for every $\epsilon>0$ we can choose $N\in\mathbb{Z}^+$ such that for all $n,m\geq N$ we have $\|a_n-a_m\|_1<\epsilon$, that is $\sum\limits_{k=1}^\infty|a_{n,k}-a_{m,k}|<\epsilon$. For each fixed $k\in\mathbb{Z}^+$, note that for $n,m\geq N$ we have $|a_{n,k}-a_{m,k}|\leq\sum\limits_{j=1}^\infty|a_{n,j}-a_{m,j}|<\epsilon$, and so the sequence $(a_{n,k})_{n\geq 1}$ is Cauchy in $\mathbb{F}$, so it converges. For each $k\in\mathbb{Z}^+$, let $b_k=\lim\limits_{n\to\infty}a_{n,k}\in\mathbb{F}$ and let $b=(b_k)_{k\geq 1}$.

We claim that $b\in\ell_1$. Since $(a_n)_{n\geq 1}$ is Cauchy, for every $\epsilon>0$ we can choose $N\in\mathbb{Z}^+$ such that for all $n,m\geq N$ we have $\|a_n-a_m\|_1<\epsilon$, that is $\sum\limits_{k=1}^\infty|a_{n,k}-a_{m,k}|<\epsilon$. By the Triangle Inequality, for $n,m\geq N$ we have $\big|\|a_n\|_1-\|a_m\|_1\big|\leq\|a_n-a_m\|_1<\epsilon$. It follows that the sequence $\big(\|a_n\|\big)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Let $M=\lim\limits_{n\to\infty}\|a_n\|_1\in\mathbb{R}$. For each fixed $K\in\mathbb{Z}^+$ we have

$$\sum_{k=1}^K|b_k|=\sum_{k=1}^K\big|\lim_{n\to\infty}a_{n,k}\big|=\lim_{n\to\infty}\sum_{k=1}^K|a_{n,k}|\leq\lim_{n\to\infty}\sum_{k=1}^\infty|a_{n,k}|=\lim_{n\to\infty}\|a_n\|_1=M.$$

Since $\sum\limits_{k=1}^K|b_k|\leq M$ for all $K\in\mathbb{Z}^+$ it follows that $\sum\limits_{k=1}^\infty|b_k|\leq M$, so $b\in\ell_1$, as claimed.

Finally, we claim that $\lim\limits_{n\to\infty}a_n=b$ in ℓ_1. Let $\epsilon>0$. Choose $N\in\mathbb{Z}^+$ such that for all $n,m\geq N$ we have $\|a_n-a_m\|_1<\epsilon$. Then for $n\geq N$ and for each $K\in\mathbb{Z}^+$ we have

$$\begin{aligned}\sum_{k=1}^K|a_{n,k}-b_k|&=\sum_{k=1}^K\big|a_{n,k}-\lim_{m\to\infty}a_{m,k}\big|=\lim_{m\to\infty}\sum_{k=1}^K|a_{n,k}-a_{m,k}|\\&\leq\lim_{m\to\infty}\sum_{k=1}^\infty|a_{n,k}-a_{m,k}|=\lim_{m\to\infty}\|a_n-a_m\|_1\leq\epsilon\end{aligned}$$

Since $\sum\limits_{k=1}^K|a_{n,k}-b_k|\leq\epsilon$ for all $K\in\mathbb{Z}^+$ it follows that $\|a_n-b\|_1=\sum\limits_{k=1}^\infty|a_{n,k}-b_k|\leq\epsilon$.

4.19 Exercise: Show that (ℓ_1, d_∞) and (ℓ_2, d_∞) are not closed in (ℓ_∞, d_∞) and so they are not complete.

4.20 Exercise: Show that the metric spaces $\big(\mathcal{C}[a,b], d_1\big)$ and $\big(\mathcal{C}[a,b], d_2\big)$ are not complete. Hint: in the case $[a,b] = [-1,1]$, consider $f_n : [-1,1] \to \mathbb{R}$ given by $f_n(x) = x^{1/2n-1}$ for $n \in \mathbb{Z}^+$. Show that if (f_n) did converge, either in $\big(\mathcal{C}[-1,1], d_1\big)$ or in $\big(\mathcal{C}[-1,1], d_2\big)$, then it would necessarily converge to a function g with $g(x) = 1$ when $x > 0$ and $g(x) = -1$ when $x < 0$, but such a function g cannot be continuous.

4.21 Definition: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For a metric space X, we define

$$\begin{aligned} \mathcal{F}(X) &= \mathcal{F}(X,\mathbb{F}) = \mathbb{F}^X = \big\{ f : X \to \mathbb{F} \big\} \\ \mathcal{B}(X) &= \mathcal{B}(X,\mathbb{F}) = \big\{ f : X \to \mathbb{F} \big|\ f \text{ is bounded} \big\} \\ \mathcal{C}(X) &= \mathcal{C}(X,\mathbb{F}) = \big\{ f : X \to \mathbb{F} \big|\ f \text{ is continuous} \big\}, \\ \mathcal{C}_b(X) &= \mathcal{C}_b(X,\mathbb{F}) = \big\{ f : X \to \mathbb{F} \big|\ f \text{ is bounded and continuous} \big\}. \end{aligned}$$

Note that $\mathcal{B}(X,\mathbb{F})$ is a normed linear space using the **supremum norm** given by

$$\|f\|_\infty = \sup_{x\in X} \big|f(x)\big|$$

and a metric space using the **supremum metric** given by $d_\infty(f,g) = \sup\limits_{x\in X} \big|f(x) - g(x)\big|$. These do not determine a well-defined norm and metric on $\mathcal{C}(X,\mathbb{F})$ since $\|f\|_\infty = \sup\limits_{x\in X} |f(x)|$ might not be finite, but they do determine a well-defined norm and metric on $\mathcal{C}_b(X,\mathbb{F})$.

4.22 Definition: For a sequence (f_n) in $\mathcal{F}(X)$ and for $g \in \mathcal{F}(X)$, we say that (f_n) **converges uniformly** to g on X, and write $f_n \to g$ uniformly on X, when for every $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|f_n(x) - g(x)| < \epsilon$ for every $n \geq m$ and every $x \in X$.

4.23 Note: For a sequence $(f_n) \in \mathcal{B}(X)$ and for $g \in \mathcal{B}(X)$, note that $|f_n(x) - g(x)| \leq \epsilon$ for every $x \in X$ if and only if $\|f_n - g\|_\infty \leq \epsilon$. It follows that $f_n \to g$ uniformly on X if and only if $f_n \to g$ in the metric space $\big(\mathcal{B}(X), d_\infty\big)$.

4.24 Theorem: *Let X be a metric space. Then the metric spaces $\big(\mathcal{B}(X), d_\infty\big)$ and $\big(\mathcal{C}_b(X), d_\infty\big)$ are complete.*

Proof: Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\big(\mathcal{B}(X), d_\infty\big)$. Note that for each $x \in X$, we have $\big|f_n(x) - f_m(x)\big| \leq \sup_{y\in X} \big|f_n(y) - f_m(y)\big| = \|f_n - f_m\|_\infty$, and so the sequence $\big(f_n(x)\big)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{F}$, so it converges. Thus we can define a function $g : X \to \mathbb{F}$ by $g(x) = \lim\limits_{n\to\infty} f_n(x)$ and then we have $f_n \to g$ pointwise in X.

We claim that $g \in \mathcal{B}(X)$, that is we claim that g is bounded. Since (f_n) is a Cauchy sequence in $\mathcal{B}(X)$, it is bounded (by Part 1 of Theorem 4.11) so we can choose $M \geq 0$ such that $\|f_n\|_\infty \leq M$ for all indices n. Then for all $x \in X$ we have $|f_n(x)| \leq \|f_n\|_\infty \leq M$ and hence $|g(x)| = \lim\limits_{n\to\infty} |f_n(x)| \leq M$. Thus g is a bounded function, that is $g \in \mathcal{B}(X)$.

We know that $f_n \to g$ pointwise on X. We must show that $f_n \to g$ uniformly on X. Let $\epsilon > 0$. Since (f_n) is Cauchy we can choose $m \in \mathbb{Z}^+$ such that $\|f_k - f_\ell\|_\infty < \epsilon$ for all $k, \ell \geq m$. Then for all $k \geq m$ and for all $x \in X$ we have

$$\big|f_k(x) - g(x)\big| = \lim_{\ell\to\infty} \big|f_k(x) - f_\ell(x)\big| \leq \epsilon.$$

It follows that $f_n \to g$ uniformly on X, that is $f_n \to g$ in the metric space $\big(\mathcal{B}(X), d_\infty\big)$. Thus $\big(\mathcal{B}(X), d_\infty\big)$ is complete.

To show that $\big(\mathcal{C}_b(X), d_\infty\big)$ is complete, it suffices (by Theorem 4.13) to show that $\mathcal{C}_b(X)$ is closed in $\mathcal{B}(X)$. Let (f_n) be a sequence in $\mathcal{C}_b(X)$ which converges in $\big(\mathcal{B}(X), d_\infty\big)$. Let $g = \lim\limits_{n\to\infty} f_n \in \mathcal{B}(X)$. We need to show that g is continuous. Let $\epsilon > 0$ and let $a \in X$. Since $f_n \to g$ in $\big(\mathcal{B}(X), d_\infty\big)$ we know that $f_n \to g$ uniformly on X, so we can choose $m \in \mathbb{Z}^+$ such that $\big|f_m(x) - g(x)\big| < \frac{\epsilon}{3}$ for all $n \geq m$ and all $x \in X$. Since f_m is continuous at a we can choose $\delta > 0$ such that for all $x \in X$ with $d(x, a) < \delta$ we have $\big|f_m(x) - f_m(a)\big| < \frac{\epsilon}{3}$. Then for all $x \in X$ with $d(x, a) < \delta$ we have

$$\big|g(x) - g(a)\big| \leq \big|g(x) - f_m(x)\big| + \big|f_m(x) - f_m(a)\big| + \big|f_m(a) - g(a)\big| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

Thus g is continuous at a. Since a was arbitrary, g is continuous on X, hence $g \in \mathcal{C}_b(X)$. By the Sequential Characterization of Closed Sets (Part 3 of Theorem 3.17) it follows that $\mathcal{C}_b(X)$ is closed in $\mathcal{B}(X)$, as required.

4.25 Corollary: *The metric space $\big(\mathcal{C}[a, b], d_\infty\big)$ is complete.*

Proof: Since every continuous function $f : [a, b] \to \mathbb{F}$ is bounded, we have $\mathcal{C}[a, b] = \mathcal{C}_b[a, b]$.

4.26 Example: For $\mathbb{F} = \mathbb{R}$, in the metric space $\big(\mathcal{C}[a, b], d_\infty\big)$, the space $\mathcal{R}[a, b]$ of Riemann integrable functions is closed, hence complete, and the spaces $\mathcal{P}[a, b]$ of polynomial functions, and $\mathcal{C}^1[a, b]$ of continuously differentiable functions, are not closed, and hence not complete.

The Completion of a Metric Space

4.27 Theorem: *(Metric Completion) Every metric space X is isometric to a dense subspace of a complete metric space.*

Proof: Let X be a metric space. Fix $a \in X$. For each $x \in X$, define $f_x : X \to \mathbb{R}$ by $f_x(t) = d(t,x) - d(t,a)$. Note that f_x is bounded since, by the Triangle Inequality, $\big|f_x(t)\big| = \big|d(x,t) - d(a,t)\big| \leq d(a,x)$. Note that f_x is continuous (indeed f_x Lipschitz continuous) because for $s,t \in X$ we have

$$\begin{aligned}\big|f_x(s) - f_x(t)\big| &= \big|d(s,x) - d(s,a) - d(t,x) + d(t,a)\big| \\ &\leq \big|d(s,x) - d(t,x)\big| + \big|d(s,a) - d(t,a)\big| \\ &\leq d(s,t) + d(s,t) = 2\,d(s,t).\end{aligned}$$

Define $F : X \to \mathcal{C}_b(X)$ by $F(x) = f_x$. We claim that F preserves distance, using the d_∞ metric on $\mathcal{C}_b(X)$. For all $x,y,t \in X$ we have

$$\big|f_x(t) - f_y(t)\big| = \big|d(x,t) - d(a,t) - d(y,t) + d(a,t)\big| = \big|d(x,t) - d(y,t)\big| \leq d(x,y)$$

hence for all $x,y \in X$ we have

$$\|f_x - f_y\|_\infty = \sup_{t \in X}\big|f_x(t) - f_y(y)\big| \leq d(x,y).$$

On the other hand, for all $x,y \in X$ we also have

$$\|f_x - f_y\|_\infty = \sup_{t \in X}\big|f_x(t) - f_y(t)\big| \geq \big|f_x(y) - f_y(y)\big| = \big|d(x,y) - d(y,y)\big| = d(x,y),$$

and so F preserves distance, as claimed. Thus X is isometric to the image $F(X) \subseteq \mathcal{C}_b(X)$, which is dense in its closure $\overline{F(X)}$, which is complete because it is a closed subspace of the complete metric space $\mathcal{C}_b(X)$.

4.28 Remark: When X is a metric space and $F : X \to \mathcal{C}_b(X)$ is the distance preserving map in the proof of the above theorem, we often identify X with its isometric image $F(X)$ and think of X as a dense subspace of the complete metric space $Y = \overline{F(X)}$. Alternatively we can do some cutting and pasting operations on sets to obtain a complete metric space Y which actually contains X as a dense subspace. Here is an outline of one possible way of constructing such a set Y. Choose a set Z which is disjoint from X and has the same cardinality as $\mathcal{C}_b(X)$ (a bit of set theory is required to prove that such a set Z exists). Choose a bijection $G : \mathcal{C}_b(X) \to Z$ and give Z the metric which makes G an isometry. Then Z is complete and the composite $H = G \circ F : X \to Z$ is distance preserving so that X is isometric to the image $H(X)$, and $H(X)$ is dense in the complete space $\overline{H(X)}$, and $\overline{H(X)}$ is disjoint from X. Then let $Y = \big(\overline{H(X)} \setminus H(X)\big) \cup X$ so that we have $X \subseteq Y$. Let $K : Y \to \overline{H(X)}$ be the bijection given by $K(x) = h(x)$ if $x \in X$ and $K(y) = y$ if $h \notin X$, and give Y the metric for which K is an isometry. Then Y is complete and X is dense in Y.

4.29 Definition: When X and Y are metric spaces with $X \subseteq Y$ such that X is dense in Y and Y is complete, we say that Y is the **metric completion** of X. The metric completion of X is unique in the sense of the following theorem.

4.30 Theorem: *(Uniqueness of the Metric Completion) Let X, Y and Z be metric spaces with Y and Z complete such that $X \subseteq Y$ with $\overline{X} = Y$ and $X \subseteq Z$ with $\overline{X} = Z$. Then there is a (unique) isometry $F : Y \to Z$ with $F(x) = x$ for all $x \in X$.*

Proof: Let $a \in Y$. Since $\overline{X} = Y$ we can choose a sequence (x_n) in X with $x_n \to a$ in Y. Then (x_n) is Cauchy in Y, hence also in X, hence also in Z. Since (x_n) is Cauchy in Z, it converges in Z, say $x_n \to b$ in Z. In order for a map $F : Y \to Z$ to be continuous with $F(x) = x$ for every $x \in X$, we must have

$$F(a) = F\big(\lim_{n\to\infty} x_n\big) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_n = b.$$

This shows that if such a map F exists, it is unique, and it must be given by the following procedure: given $a \in Y$ we choose a sequence (x_n) in X with $x_n \to a$ and then we define $F(a) = \lim\limits_{n\to\infty} x_n \in Z$.

We claim that the above procedure does determine a well-defined map whose value $F(a)$ does not depend on the choice of the sequence (x_n). Let $a \in Y$ and let (x_n) and (y_n) be two sequences in X with $x_n \to a$ and $y_n \to a$ in Y. Let $b = \lim\limits_{n\to\infty} x_n$ in Z and let $c = \lim\limits_{n\to\infty}$ in Z. We need to show that $b = c$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that for all indices $n \geq m$ we have $d_Y(x_n, a) < \frac{\epsilon}{4}$, $d_Y(y_n, a) < \frac{\epsilon}{4}$, $d_Z(x_n, b) < \frac{\epsilon}{4}$. and $d_Z(y_n, c) < \frac{\epsilon}{4}$. Then since $d_Z(x_n, y_n) = d_X(x_n, y_n) = d_Y(x_n, y_n)$ we have

$$\begin{aligned} d_Z(b,c) &\leq d_Z(b,x_n) + d_Z(x_n,y_n) + d_Z(y_n,c) \\ &= d_Z(b,x_n) + d_Y(x_n,y_n) + d_Z(y_n,c) \\ &\leq d_Z(b,x_n) + d_Y(x_n,a) + d_Y(a,y_n) + d_Z(y_n,c) \\ &< \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \epsilon. \end{aligned}$$

Since $d_Z(b,c) < \epsilon$ for every $\epsilon > 0$ we must have $d_Z(b, c{=}0$ hence $b = c$, as required.

Note that F is bijective with its inverse G given by the same construction: given $c \in Z$ we choose a sequence (x_n) in X with $x_n \to b$ in Z and define $G(c) = b = \lim\limits_{n\to\infty} x_n$ in Y.

It remains to prove that F preserves distance. Let $a, b \in Y$. Chooose sequences (x_n) and (y_n) in X with $x_n \to a$ and $y_n \to b$ in Y. Let $c, d \in Z$ with $x_n \to c$ and $y_n \to d$ in Z. We need to show that $d_Y(a,b) = d_Z(c,d)$. Since

$$\begin{aligned} d_Y(a,b) &\leq d_Y(a,x_n) + d_Y(x_n,y_n) + d_Y(y_n,b) \text{ , and} \\ d_Y(x_n,y_n) &\leq d_Y(x_n,a) + d_Y(a,b) + d_Y(b,y_n) \end{aligned}$$

it follows that

$$\big|d_Y(a,b) - d_Y(x_n,y_n)\big| \leq d_Y(a,x_n) + d_Y(y_n,b).$$

Taking the limit as $n \to \infty$ gives $\big|d_Y(a,b) - \lim\limits_{n\to\infty} d_Y(x_n,y_n)\big| = 0$ so that

$$d_Y(a,b) = \lim_{n\to\infty} d_Y(x_n,y_n) = \lim_{n\to\infty} d_X(x_n,y_n).$$

Similarly, we have $d_Z(c,d) = \lim\limits_{n\to\infty} d_X(x_n,y_n)$ and hence $d_Y(a,b) = d_Z(c,d)$, as required.