PMATH 351 Real Analysis, Exercises for Chapter 5: Connectedness and Compactness

1: (a) Prove Theorem 5.18: let X be a metric space (or topological space) and prove that the path components of X are path connected, and that every path connected subset of X is contained in one of the path components.

(b) Let X be a metric space and let $A, B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$. Show that if A is connected then so is B.

(c) Let X be a metric space. Show that the connected components of X are closed.

2: For each of the following sets A in $\mathbb{R}^n$ (using its standard metric), determine whether A is complete, whether A is compact, and whether A is connected.

(a) $A = \left\{ x \in \mathbb{R}^3 \,\middle|\, \|x\| = \frac{n-1}{n^2} \text{ for some } n \in \mathbb{Z}^+ \right\}$.

(b) Let $A = \left\{ (u,v,w,x,y,z) \in \mathbb{R}^6 \,\middle|\, \operatorname{rank} \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} < 2 \right\}$.

3: For each of the following sets A, determine whether A is complete and whether A is compact.

(a) $A = \left\{ a = (a_k)_{k \geq 1} \in \mathbb{R}^\infty \,\middle|\, \|a\|_\infty \leq 1 \right\} \subseteq \mathbb{R}^\infty \subseteq \ell_\infty(\mathbb{R})$, using the metric d_∞.

(b) $A = \left\{ a = (a_k)_{k \geq 1} \in \ell_1(\mathbb{R}) \,\middle|\, \sum\limits_{k=1}^{\infty} a_k = 0 \right\} \subseteq \ell_1(\mathbb{R})$, using the metric d_1.

(c) $A = \left\{ f \in \mathcal{C}([0,1],\mathbb{R}) \,\middle|\, |f(x)| \leq \frac{1}{x} \text{ for all } x \in (0,1) \right\} \subseteq \mathcal{C}([0,1],\mathbb{R})$, using the metric d_∞.

4: The theorems about connectedness in the lecture notes are stated for metric spaces. They also apply, more generally, to topological spaces with minor alterations in the proofs, with a slight change in Definition 5.1. Let X be a topological space and let $P \subseteq X$. For sets $A, B \subseteq X$ (not necessarily open sets), we say that A and B **separate** P in X when

$$A \cap P \neq \emptyset\,,\ B \cap P \neq \emptyset\,,\ A \cap \overline{B} = B \cap \overline{A} = \emptyset\,,\ P \subseteq A \cup B\,.$$

Recall that we say that P is **connected** when P is not equal to the union of two disjoint nonempty open subsets $U, V \subseteq P$ (or equivalently when the only subsets of P which are both open and closed in P are the sets $\emptyset$ and P), and otherwise we say that P is **disconnected**.

(a) Let X be a topological space and let $A \subseteq P \subseteq X$. Let $\overline{A} = \mathrm{Cl}_X(A)$ be the closure of A in X and let $\mathrm{Cl}_P(A)$ be the closure of A in P. Show that $\mathrm{Cl}_P(A) = \mathrm{Cl}_X(A) \cap P$.

(b) Let X be a topological space and let $P \subseteq X$. Show that P is disconnected if and only if there exist sets $A, B \subseteq X$ which separate P in X, as defined above.

(c) Let X be a metric space and let $P \subseteq X$. Show that there exist sets $A, B \subseteq X$ which separate P, as defined above, if and only if there exist open sets $U, V \subseteq X$ which separate P as in Definition 5.1, that is

$$U \cap P \neq \emptyset\,,\ V \cap P \neq \emptyset\,,\ U \cap P = \emptyset\,,\ P \subseteq U \cup V\,.$$

6: (a) Prove Theorem 5.37: let X and Y be metric spaces and let $f : X \to Y$. Show that if X is compact and f is continuous then f is uniformly continuous.

(b) Let X be a metric space. Show that if X is compact then X must be separable.

(c) Let U be a non-trivial finite-dimensional vector space over $\mathbb{R}$. Show that there does not exist a norm on U which makes U compact, but there does exist a metric on U which makes U compact.

7: (a) Define a metric on $\mathbb{R}$ by $d(x,y) = \frac{|x-y|}{1+|x-y|}$ (you do not need to prove that d is a metric). Show that $(\mathbb{R}, d)$ is bounded but not totally bounded (hence not compact).

(b) Let $X = 2^{\mathbb{Z}^+}$ be the space of binary sequences $(x_k)_{k \geq 1}$ with each $x_k \in \{0,1\}$. Define a metric on X by $d(x,y) = \sum\limits_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k}$ (you do not need to prove that d is a metric). Show that X is complete and totally bounded (hence compact).

8: (a) Let $B = \overline{B}(0,1) = \left\{ x \in \ell_2(\mathbb{R}) \,\middle|\, ||x||_2 \leq 1 \right\} \subseteq \ell_2(\mathbb{R})$. Show that B is not compact in $(\ell_2(\mathbb{R}), d_2)$.

(b) Let $r_k \geq 0$ for all $k \in \mathbb{Z}^+$, and let $S = \left\{ x \in \ell_2(\mathbb{R}) \,\middle|\, |x_k| \leq r_k \text{ for all } k \in \mathbb{Z}^+ \right\} \subseteq \ell_2(\mathbb{R})$. Show that S is compact in $(\ell_2(\mathbb{R}), d_2)$ if and only if $\sum\limits_{k=1}^{\infty} {r_k}^2$ converges in $\mathbb{R}$.