PMATH 351 Real Analysis, Exercises for Chapter 6: Applications

In these exercises, when X is a metric space, we write $\mathcal{C}(X)$ to denote $\big(\mathcal{C}(X,\mathbb{R}), d_\infty\big)$.

1: (a) Find an example of a function $f:\mathbb{R}\to\mathbb{R}$ such that $\big|f(y)-f(x)\big| < |y-x|$ for all $x,y\in\mathbb{R}$ with $x\neq y$, but f has no fixed point in $\mathbb{R}$.

(b) The polynomial $p(x)=x^3-3x+1$ has a unique root in $\left[0,\frac{1}{2}\right]$. Approximate this root using the Banach Fixed Point Theorem as follows: Let $f(x)=\frac{1}{3}(x^3+1)$. Show that $f:\left[0,\frac{1}{2}\right]\to\left[0,\frac{1}{2}\right]$ is a contraction map whose unique fixed point is the desired root of p. Approximate the root by using a calculator to find x_5 where $x_0=0$ and $x_{n+1}=f(x_n)$.

2: (a) Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=\displaystyle\int_0^x f(t)\,dt$. Show that F is not a contraction map but $F^2=F\circ F$ is.

(b) Use the Banach Fixed Point Theorem to show that there exists a unique function $f\in\mathcal{C}[0,1]$ such that $f(x)=x+\displaystyle\int_0^x t\,f(t)\,dt$ for all $x\in[0,1]$.

3: Solve the differential equation $y'=1+x^2y$ with $y(0)=0$ in the interval $[-1,1]$ using the following method: Define $F:\mathcal{C}[-1,1]\to\mathcal{C}[-1,1]$ by $F(f)(x)=x+\displaystyle\int_0^x t^2 f(t)\,dt$. Show that F is a contraction map (using the supremum norm) whose unique fixed point is the desired solution. Express the solution as a power series by finding a formula for $f_n(x)$ where $f_0(x)=0$ and $f_{n+1}(x)=F(f_n)(x)$.

4: (a) Let $A=\left\{\sum\limits_{k=1}^{n} f_k(x)g_k(y)\,\middle|\, n\in\mathbb{Z}^+, f_k,g_k\in\mathcal{C}[0,1]\right\}$. Show that A is dense in $\mathcal{C}\big([0,1]\times[0,1]\big)$.

(b) Let $A=\left\{\sum\limits_{k=0}^{n}\big(a_k\sin(kx)+b_k\cos(kx)\big)\,\middle|\, 0\leq n\in\mathbb{Z}, a_k,b_k\in\mathbb{R}\right\}$. Show that A is dense in $\mathcal{C}[0,\pi]$ but A is not dense in $\mathcal{C}[0,2\pi]$.

5: (a) Let $f\in\mathcal{C}[0,1]$. Suppose that $\displaystyle\int_0^1 f(x)\,dx=0$ and $\displaystyle\int_0^1 x^{12+3n}f(x)\,dx=0$ for all $n\in\mathbb{Z}^+$. Use the Stone-Weierstrass Theorem to show that $f(x)=0$ for all $x\in[0,1]$.

(b) Show that there does exist $0\neq f\in\mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^2 x^{2n}f(x)\,dx=0$ for all $0\leq n\in\mathbb{Z}$ but there does not exist $0\neq f\in\mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^2 x^{3n}f(x)\,dx=0$ for all $0\leq n\in\mathbb{Z}$.

6: (a) For $n\in\mathbb{Z}^+$, define $f_n:[0,2\pi]\to\mathbb{R}$ by $f_n(x)=(\sin x)^n$. Determine whether the set $A=\left\{f_n\,\middle|\,n\in\mathbb{Z}^+\right\}$ is equicontinuous.

(b) Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be continuous. For each $y\in[0,1]$ define $f_y:[0,1]\to\mathbb{R}$ by $f_y(x)=f(x,y)$. Show that the set $A=\left\{f_y\,\middle|\,y\in[0,1]\right\}$ is compact in $\mathcal{C}[0,1]$.

(c) Show that the closed unit ball $\overline{B}(0,1)=\left\{f\in\mathcal{C}[0,1]\,\middle|\,\|f\|_\infty\leq 1\right\}$ cannot be covered by a countable set of compact sets in $\mathcal{C}[0,1]$.