Chapter 3. Geodesic Curvature and the Gauss-Bonnet Theorem

Geodesic Curvature and Geodesics

3.1 Definition: Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface, let $\alpha : I \subseteq \mathbb{R} \to U$ be a smooth regular curve, and let $\gamma(t) = \sigma(\alpha(t))$. Reparametrize γ by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ and $\delta(s) = \gamma(t(s))$ where $s(t) = \int_a^t |\gamma'(r)|\,dr$. Let $T(s) = \delta'(s)$, let $N(s) = n\big(\beta(s)\big)$, and let $M(s) = N(s)\times T(s)$ so $\{T, M, N\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ with T and M spanning the surface's tangent space. Since $1 = \delta' \bullet \delta'$, we have $\delta'' \bullet \delta' = 0$, that is $\delta'' \bullet T = 0$, so δ'' is in the span of N and M. The curvature vector $\delta''(s)$ can be decomposed into its normal and tangential components as

$$\delta'' = (\delta'' \bullet N)N + (\delta'' \bullet M)M.$$

We have already studied the normal component $\delta'' \bullet N$: it is the directional curvature of σ in the direction of δ' (it does not depend on the shape of the curve δ, but only on the shape of the surface σ and the direction of δ'). The tangential component $\delta'' \bullet M$ does depend on the shape of the curve. The **geodesic curvature** of β on σ at s is given by

$$k_g = k_g(s) = k_g(\beta)(s) = \delta''(s) \bullet M(s),$$

and the **geodesic curvature** of α on σ at t is given by $k_g = k_g(t) = k_g(\alpha)(t) = k_g(\beta)(s(t))$.

3.2 Theorem: *Geodesic curvature is an intrinsic property. Indeed for σ, α, β, γ and δ as above, if we write $\beta(s) = \big(u(s), v(s)\big)$ then*

$$k_g = \sqrt{\det g}\,\Big(\big(v'' + \Gamma^2_{11}(u')^2 + 2\,\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2\big)u' - \big(u'' + \Gamma^1_{11}(u')^2 + 2\,\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2\big)v'\Big).$$

Proof: We have

$$k_g = \delta'' \bullet M = \delta'' \bullet (N \times T) = \delta'' \bullet (N \times \delta') = \det(\delta', \delta'', N)$$

where $\delta(s) = \sigma(\beta(s)) = \sigma\big(u(s), v(s)\big)$ and $N(s) = n(\beta(s)) = n\big(u(s), v(s)\big)$. By the Gauss-Weingarten equations, we have

$$\begin{aligned}
\delta' &= \sigma_u u' + \sigma_v v' \\
\delta'' &= \sigma_{uu}(u')^2 + \sigma_{uv}u'v' + \sigma_u u'' + \sigma_{uv}u'v' + \sigma_{vv}(v')^2 + \sigma_v v'' \\
&= (\Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + h_{11}n)(u')^2 + 2(\Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + h_{12}n)u'v' \\
&\quad + (\Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + h_{22}n)(v')^2 + \sigma_u u'' + \sigma_v v'' \\
&= \sigma_u\big(u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2\big) \\
&\quad + \sigma_v\big(v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2\big) \\
&\quad + n\big(h_{11}(u')^2 + 2h_{12}u'v' + h_{22}(v')^2\big).
\end{aligned}$$

Since the determinant of a matrix is an alternating linear function of its columns, the above formulas for δ' and δ'' give

$$\begin{aligned}
k_g &= \det(\delta', \delta'', N) \\
&= \det(\sigma_u, \sigma_v, n)\Big(u'\big(v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2\big) \\
&\qquad\qquad - v'\big(\Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2\big)\Big).
\end{aligned}$$

Finally, we note that $\det(\sigma_u, \sigma_v, n) = (\sigma_u \times \sigma_v) \bullet \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = |\sigma_u \times \sigma_v| = \sqrt{\det g}$ (with both $|\sigma_u \times \sigma_v|$ and $\sqrt{\det g}$ being equal to the area of the parallelogram on σ_u and σ_v).

3.3 Remark: We shall define a geodesic on a surface, and our intent is that a geodesic should be a curve which is locally of minimum possible arclength. Before stating the formal definition, let us provide some motivation (be warned that this motivational discussion is about two pages long). Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$, let $\alpha : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ be a smooth regular curve, and let $\gamma(t) = \sigma(\alpha(t))$. The length of γ on $[t_1, t_2] \subseteq I$ in $\mathbb{R}^3$ (or the length of α on $[t_1, t_2]$ with respect to the Riemanninan metric g) is given by

$$L_\gamma\big([t_1, t_2]\big) = \int_{t_1}^{t_2} \big|\gamma'(t)\big|\, dt = \int_{t_1}^{t_2} \sqrt{\alpha'(t) g(\alpha(t))\alpha'(t)}\ dt\,.$$

Define the **energy** of γ on $[t_1, t_2]$ in (or the energy of α on $[t_1, t_2]$ with respect to g) to be

$$E_\gamma\big([t_1, t_2]\big) = \int_{t_1}^{t_2} \big|\gamma'(t)\big|^2 dt = \int_{t_1}^{t_2} \alpha'(t)^T g(\alpha(t))\alpha'(t)\ dt.$$

By the Cauchy-Schwarz inequality, applied to the inner product on continuous functions given by $\langle f, g\rangle = \int_a^b f(t)g(t)\, dt$, when f is continuous with $f(t) \geq 0$ for all $t \in [a, b]$ we have

$$\left(\int_a^b \sqrt{f(t)}\, dt\right)^2 = \Big|\big\langle 1, \sqrt{f}\,\big\rangle\Big|^2 \leq \big\|1\big\|^2\big\|\sqrt{f}\big\|^2 = (b-a)\int_a^b f(t)\, dt$$

with equality if and only if $\big\{\sqrt{f}, 1\big\}$ is linearly dependent, that is if and only if f is a constant function. In particular $\big($taking $f(t) = |\gamma'(t)|^2 = \alpha'(t)^T g(\alpha(t))\alpha'(t)\big)$, we have

$$L_\gamma\big([t_1, t_2]\big)^2 \leq (t_2 - t_1)E_\gamma\big([t_1, t_2]\big)$$

with equality if and only if $|\gamma'(t)|$ is constant. If we reparametrize γ by arclength by letting $\beta(s) = \alpha(t(s))$ and $\delta(s) = \sigma(\beta(s)) = \gamma(t(s))$ where $s(t) = \int_a^t |\gamma'(r)|\, dr$, then $|\delta'(s)|$ is constant (indeed $|\delta'(s)| = 1$ for all s) so we have

$$L_\delta\big([s_1, s_2]\big)^2 = (s_2 - s_1)E_\delta\big([s_1, s_2]\big),$$

and so, to minimize the arclength we can minimize the energy.

To find a curve $\beta(s)$, parametrized by arclength with respect to g, which is locally of minimum energy, we use the methods of calculus of variations, which we now summarize for students who are unfamiliar with this branch of mathematics. A **functional** is a function whose domain is a set of functions. Given a smooth function $L = L(s, u, v, u', v')$, define a functional F by

$$F(\beta) = \int_a^b L\big(s, u(s), v(s), u'(s), v'(s)\big)\ ds$$

where $\beta : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ is given by $\beta(s) = \big(u(s), v(s)\big)$. The function L is called the **Langrangian** for the functional F. In order for the functional F to have a local minimum (or maximum) at β, the functions $\phi_\epsilon(t) = F\big(\beta + t(\epsilon, 0)\big)$ and $\psi_\epsilon(t) = F\big(\beta + t(0, \epsilon)\big)$ must both have a local minimum (or maximum) value at $t = 0$ for any smooth function $\epsilon : [a, b] \to \mathbb{R}$ with $\epsilon(a) = \epsilon(b) = 0$, so we must have $\phi_\epsilon'(0) = 0$ and $\psi_\epsilon'(0)$ for any such ϵ.

Using Integration by Parts, we have

$$
\begin{aligned}
\phi_\epsilon{}'(t) &= \tfrac{d}{dt}F(\beta+t\epsilon) = \int_a^b \tfrac{d}{dt}L\big(s, u(s)+t\epsilon(s), v(s), u'(s)+t\epsilon'(s), v'(s)\big)\,ds \\
&= \int_a^b \tfrac{\partial L}{\partial u}\epsilon + \tfrac{\partial L}{\partial u'}\epsilon'\ ds = \int_a^b \tfrac{\partial L}{\partial u}\ \epsilon\ ds + \Big[\tfrac{\partial L}{\partial u'}\ \epsilon\Big]_a^b - \int_a^b \tfrac{d}{ds}\tfrac{\partial L}{\partial u'}\ \epsilon\ ds, \\
\phi_\epsilon{}'(0) &= \int_a^b \big(\tfrac{\partial L}{\partial u} - \tfrac{d}{ds}\tfrac{\partial L}{\partial u'}\big)\epsilon\ ds.
\end{aligned}
$$

In order for the final integral to vanish for every such ϵ, we must have $\frac{\partial L}{\partial u} - \frac{d}{ds}\frac{\partial L}{\partial u'} = 0$. Similarly, in order to have $\psi_\epsilon{}'(0)$ vanish for every such ϵ, we must have $\frac{\partial L}{\partial v} - \frac{d}{ds}\frac{\partial L}{\partial v'} = 0$. The equations

$$
\tfrac{\partial L}{\partial u} - \tfrac{d}{ds}\tfrac{\partial L}{\partial u'} = 0 \quad , \quad \tfrac{\partial L}{\partial v} - \tfrac{d}{ds}\tfrac{\partial L}{\partial v'} = 0
$$

are called the **Euler-Lagrange equations** for the functional F with Lagrangian L, and any function β at which the functional F has a local minimum (or maximum) must satisfy these equations.

Returning to our problem of defining a geodesic to be a curve which locally minimizes arclegth, we are interested in finding a curve $\beta = \beta(s) = \big(u(s), v(s)\big)$, such that the curve $\delta(s) = \sigma(\beta(s))$ is parametrized by arclength, and β is a local minimum for the **energy functional**

$$
F(\beta) = E_\delta\big([a,b]\big) = \int_a^b \beta'(s)^T g(\beta(s))\beta'(s)\ ds\,,
$$

which is the functional with Lagrangian

$$
\begin{aligned}
L &= L(s,u,v,u',v') = (u',v')g(u,v)\big(\begin{smallmatrix} u' \\ v' \end{smallmatrix}\big) \\
&= g_{11}(u,v)\,(u')^2 + 2g_{12}(u,v)\,u'v' + g_{22}(u,v)\,(v')^2.
\end{aligned}
$$

For this particular problem, we expect (intuitively) that all solutions to the Euler-Lagrange equations will yield local minima $\big($partly because given $t > 0$ we can choose ϵ to be a rapidly oscillating function so that $\phi_\epsilon(t)$ and $\psi_\epsilon(t)$ are arbitrarily large, so $\phi_\epsilon(t)$ and $\psi_\epsilon(t)$ never have local maxima at $t = 0\big)$, so we shall define a geodesic to be a curve, parametrized by arclength with respect to g, which satisfies the Euler-Lagrange equations for this functional. We remark that it is possible to prove rigorously that all solutions to the Euler-Lagrange equations do locally minimize arclength, by using some additional techniques from calculus of variations (by calculating the so called *second variation*).

Our final chore, before stating the formal definition of a geodesic, it to calculate the Euler-Lagrange equations for the above energy functional. We have

$$
\begin{aligned}
\tfrac{\partial L}{\partial u} &= (g_{11})_u(u')^2 + 2(g_{12})_u u'v' + (g_{22})_u(v')^2 \\
\tfrac{d}{ds}\tfrac{\partial L}{\partial u'} &= \tfrac{d}{ds}\big(2\,g_{11}u' + 2g_{12}v'\big) = \tfrac{d}{ds}\big(2\,g_{11}\big(u(s),v(s)\big)u'(s) + 2\,g_{12}\big(u(s),v(s)\big)v'(s)\big) \\
&= 2\Big((g_{11})_u(u')^2 + (g_{11})_v u'v' + g_{11}u'' + (g_{12})_u u'v' + (g_{12})_v(v')^2 + g_{12}v''\Big)
\end{aligned}
$$

and so we have $\frac{\partial L}{\partial u} = \frac{d}{ds}\frac{\partial L}{\partial u'}$ if and only if

$$
2g_{11}u'' + 2g_{12}v'' + (g_{11})_u(u')^2 + 2(g_{11})_v u'v' + \big(2(g_{12})_v - (g_{22})_u\big)(v')^2 = 0\,.
$$

A similar calculation shows that $\frac{\partial L}{\partial v} - \frac{d}{dt}\frac{\partial L}{\partial v'} = 0$ if and only if

$$
2g_{12}u'' + 2g_{22}v'' + \big(2(g_{12})_u - (g_{11})_v\big)(u')^2 + 2(g_{22})_u u'v' + (g_{22})_v(v')^2 = 0\,.
$$

Thus both of the Euler-Lagrange equations hold if and only if

$$g\begin{pmatrix} u'' \\ v'' \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} (g_{11})_u(u')^2 + 2(g_{11})_v u'v' + \big(2(g_{12})_v - (g_{22})_u\big)(v')^2 \\ \big(2(g_{12})_u - (g_{11})_v\big)(u')^2 + 2(g_{22})_u u'v' + (g_{22})_v(v')^2 \end{pmatrix}$$

that is if and only if

$$\begin{aligned}\begin{pmatrix} u'' \\ v'' \end{pmatrix} &= -\tfrac{1}{2}g^{-1}\begin{pmatrix} (g_{11})_u(u')^2 + 2(g_{11})_v u'v' + \big(2(g_{12})_v - (g_{22})_u\big)(v')^2 \\ \big(2(g_{12})_u - (g_{11})_v\big)(u')^2 + 2(g_{22})_u u'v' + (g_{22})_v(v')^2 \end{pmatrix} \\ &= -\begin{pmatrix} \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 \\ \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 \end{pmatrix}\end{aligned}$$

3.4 Definition: Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$. A **geodesic** on σ is a smooth regular curve $\beta : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$, parametrized by arclength with respect to g $\big($meaning that the curve $\delta(s) = \sigma(\beta(s))$ satisfies $|\delta'(s)| = 1$ for all $s\big)$, which satisfies the Euler-Lagrange equations for the **energy functional**

$$F(\beta) = E_\delta\big([a,b]\big) = \int_a^b \beta'(s)^T g(\beta(s))\beta'(s)\, ds\,,$$

which is equivalent to saying that for $\beta(s) = \big(u(s), v(s)\big)$ we have

$$\begin{aligned} u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0\,, \\ v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0\,. \end{aligned}$$

The above differential equations are called the **geodesic equations**.

3.5 Theorem: *Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface, let $\beta : I \subseteq \mathbb{R} \to U$ be a smooth regular curve (not necessarily parametrized by arclength), and let $\delta(s) = \sigma(\beta(s))$.*

(1) $\delta''(s)$ is in the direction of $N(s) = n(\beta(s))$ for all s if and only if β satisfies the geodesic equations and, in this case, $|\delta'|$ is constant.

(2) When $|\delta'(s)| = 1$ for all s, β is a geodesic if and only if k_g is identically zero.

Proof: From the formula

$$\begin{aligned}\delta'' &= \sigma_u\big(u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2\big) \\ &\quad + \sigma_v\big(v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2\big) \\ &\quad + n\big(h_{11}(u')^2 + 2h_{12}u'v' + h_{22}(v')^2\big),\end{aligned}$$

which was derived in the proof of Theorem 3.2, it is apparent that $\delta''(s)$ is in the span of $N(s) = n(\beta(s))$ for all s if and only if β satisfies the geodesic equations. And in this case, since δ'' is orthogonal to the tangent space, we have $\delta'' \bullet \delta' = 0$, that is $\frac{d}{ds}\delta' \bullet \delta' = 0$, and so $|\delta'|$ is constant, proving Part 1.

To prove Part 2, suppose that $|\delta'(s)| = 1$ for all s. If β satisfies the geodesic equations then $k_g(s) = 0$ for all s by the formula for k_g in Theorem 3.2. Suppose, on the other hand, that k_g is identically zero. From the definition of k_g, we have $\delta''(s) \bullet M(s) = 0$ for all s. Since $|\delta'(s)| = 1$ for all s we have $\delta''(s) \bullet \delta'(s) = 0$, that is $\delta'(s) \bullet T(s) = 0$ for all s. Since $\delta''(s)$ is orthogonal to both $M(s)$ and $T(s)$, it is in the direction of $N(s)$ for all s, hence β satisfies the geodesic equations by Part 1, so β is a geodesic.

3.6 Corollary: *Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface. Given a point $p \in U$ and a unit vector $A \in \mathbb{R}^2$, there is a unique geodesic $\beta : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ with $0 \in I$, $\beta(0) = p$ and $\frac{\beta'(0)}{|\beta'(0)|} = A$, where I is the maximal open interval with $\beta(s) \in U$ for all $s \in I$.*

Proof: This follows from existence and uniqueness theorems for differential equations.

Orthogonal Coordinates

3.7 Theorem: *(Orthogonal Coordinates) Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$. For every $p \in U$ there exists an open set $U_p \subseteq U$ with $p \in U_p$ and a smooth regular change of coordnates $\phi : U_p \to V_p$ with $\phi(p) = p$ such that, for the corresponding surface given by $\rho(s,t) = \sigma\big(\phi^{-1}(s,t)\big)$, the matrix $g_\rho(s,t)$ is diagonal for all s,t.*

Proof: We sketch a proof, making use of existence and uniqueness theorems for differential equations. Write $\psi(s,t) = \phi^{-1}(s,t) = \big(u(s,t), v(s,t)\big)$ so that $\rho(s,t) = \sigma\big(u(s,t), v(s,t)\big)$. Choose $v = v(s,t) = t$ so that $v_s = 0$ and $v_t = 1$. Then

$$\begin{aligned}\rho_s &= \sigma_u u_s + \sigma_v v_s = \sigma_u u_s \\ \rho_t &= \sigma_u u_t + \sigma_v v_t = \sigma_u u_t + \sigma_v \\ \rho_s \bullet \rho_t &= (\sigma_u \bullet \sigma_u) u_s u_t + (\sigma_u \bullet \sigma_v) u_s\end{aligned}$$

that is $(g_\rho)_{12} = \big((g_\sigma)_{11} u_t + (g_\sigma)_{12}\big) u_s$. So to get $(g_\rho)_{12} = 0$ we can choose $u = u(s,t)$ so that $u_t = -\frac{(g_\sigma)_{12}}{(g_\sigma)_{11}}$. To be more explicit, if $p = (a,b)$ then for each s we let $u_s = u_s(t)$ be the (unique) solution to the ordinary differential equation $u_s{}' = -\frac{(g_\sigma)_{12}(u_s,t)}{(g_\sigma)_{11}(u_s,t)}$ with the initial condition $u_s(b) = a$, then let $u(s,t) = u_s(t)$. Note that the solution exists locally, and for the resulting surface ρ, the derivative matrix $D\rho$ will be of rank 2 locally, near p. In addition, since the functions $(g_\sigma)_{12}$ and $(g_\sigma)_{11}$ are smooth, the solution is smooth.

3.8 Definition: When $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a regular smooth surface for which the first fundamental form $g = g(u,v)$ is diagonal for all $(u,v) \in U$, we say that the coordinates (u,v) are **orthogonal coordinates**.

3.9 Remark: In fact, Gauss proved a stronger version of the above theorem. He showed that given any smooth regular surface σ in $\mathbb{R}^3$, in a neighbourhood of each point $p \in U$, there exists a smooth regular change of coordinates such that $g_\rho(s,t)$ is a multiple of the identity matrix for all (s,t). Such coordinates (s,t) are called **isothermal coordinates**.

3.10 Theorem: *(Gaussian Curvature in Orthogonal Coordinates) Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$ for which $g = g(u,v)$ is diagonal for all $(u,v) \in U$. Then the Gaussian curvature $K = K(u,v)$ is given by*

$$K = \frac{-1}{2\sqrt{g_{11}g_{22}}}\left(\frac{\partial}{\partial u}\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}} + \frac{\partial}{\partial v}\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right).$$

Proof: We sketch the proof. From the definition of the Christoffel symbols given in Theorem 2.20 (the Gauss-Weingarten equations), we have

$$\Gamma = \begin{pmatrix}\Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\frac{1}{g_{11}} & 0 \\ 0 & \frac{1}{g_{22}}\end{pmatrix}\begin{pmatrix}(g_{11})_u & (g_{11})_v & -(g_{22})_u \\ -(g_{11})_v & (g_{22})_u & (g_{22})_v\end{pmatrix}.$$

Put these values into the formula for K given in the proof of Theorem 2.23 (Gauss' Theorema Egregium) to get

$$g_{11}g_{22}K = g_{11}\Big((\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{11}^1\Gamma_{22}^1 + \Gamma_{12}^1\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{12}^1 - \Gamma_{22}^1\Gamma_{12}^2\Big),$$

$$K = \tfrac{1}{g_{22}}\Big(\tfrac{\partial}{\partial u}\big(-\tfrac{(g_{22})_u}{2\,g_{11}}\big) - \tfrac{\partial}{\partial v}\big(\tfrac{(g_{11})_v}{2\,g_{11}}\big) - \tfrac{(g_{11})_u(g_{22})_u}{4\,g_{11}{}^2} + \tfrac{(g_{11})_v(g_{22})_v}{4\,g_{11}g_{22}} - \tfrac{{(g_{11})_v}^2}{4\,g_{11}{}^2} - \tfrac{{(g_{22})_u}^2}{4\,g_{11}g_{22}}\Big).$$

To complete the proof, take the derivatives, expand and simplify, and do the same to the expression on the right hand side in the statement of the theorem.

3.11 Theorem: *(Geodesic Curvature of Coordinate Lines in Orthogonal Coordinates) Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$ for which $g = g(u,v)$ is diagonal for all $(u,v) \in U$. At the point $p = (a,b) \in U$, the geodesic curvatures of the coordinate lines $v = b$ and $u = a$ (that is the curves given by $\alpha = (u(s), b)$ with $u(0) = a$ and $u'(s) > 0$, and $\beta(s) = (a, v(s))$ with $v(0) = b$ and $v'(s) > 0$, both parametrized by arclength with respect to g) are given by*

$$k_1 = k_g^{v=b} = k_g(\alpha) = -\frac{(g_{11})_v}{2\, g_{11}\sqrt{g_{22}}} \quad , \quad k_2 = k_g^{u=a} = k_g(\beta) = \frac{(g_{22})_u}{2\, g_{22}\sqrt{g_{11}}} \,.$$

Proof: For any curve $\beta(s) = \big(u(s), v(s)\big)$, parametrized by arclength with respect to g, from the formula in Theorem 3.2, we have

$$k_g = \sqrt{\det g}\,\Big(\big(v'' + \Gamma^2_{11}(u')^2 + 2\,\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2\big)u' - \big(u'' + \Gamma^1_{11}(u')^2 + 2\,\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2\big)v'\Big).$$

For the particular curve $\beta(s) = \big(a, v(s)\big)$, where $u(s) = a$, we have $u' = 0$ and $u'' = 0$, so this simplifies to

$$k_2 = k_g^{u=a} = -\sqrt{\det g}\;\Gamma^1_{22}\,(v')^3.$$

Because β is parametrized by arclength (with respect to g) and $u' = 0$ we have

$$1 = \beta'(s)^T g(\beta(s))\beta'(s) = g_{11}(u')^2 + 2g_{12}u'v' + g_{22}(v')^2 = g_{22}(v')^2$$

so that $v' = \frac{1}{\sqrt{g_{22}}}$. Because $g_{12} = 0$ we have $\det g = g_{11}g_{22}$ and we have $\Gamma^1_{22} = -\frac{(g_{22})_u}{2\,g_{11}}$ (as in the proof of Theorem 3.10), and so

$$k_2 = k_g^{u=a} = -\sqrt{\det g}\;\Gamma^1_{22}\,(v')^3 = -\sqrt{g_{11}g_{22}}\,\Big(-\tfrac{(g_{22})_u}{2\,g_{11}}\Big)\Big(\tfrac{1}{\sqrt{g_{22}}}\Big)^3 = \frac{(g_{22})_u}{2\,g_{22}\sqrt{g_{11}}}\,,$$

as required. The calculation of $k_1 = k_g^{v=b}$ is similar.

3.12 Note: Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$ for which $g = g(u,v)$ is diagonal for all $(u,v) \in U$. Let $\beta : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ be a smooth regular curve which is parametrized by arclength with respect to g, so we have $|\delta'(s)| = 1$ for all $s \in I$, where $\delta(s) = \sigma(\beta(s))$. Since $\sigma_u \cdot \sigma_v = g_{12} = 0$ so that $|\sigma \times \sigma_v| = |\sigma_u|\,|\sigma_v|$, we have $n = \frac{\sigma_u}{|\sigma_u} \times \frac{\sigma_v}{|\sigma_v|}$ so that $\big\{\frac{\sigma_u}{|\sigma_u|}, \frac{\sigma_v}{|\sigma_v|}, n\big\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$. Since $\delta(s) = \sigma(\beta(s)) = \sigma(u(s), v(s))$, we have $\delta' = \sigma_u u' + \sigma_v v'$ so that δ' is in the span of σ_u and σ_v, and so $\delta' = \big(\delta' \cdot \frac{\sigma_u}{|\sigma_u|}\big)\frac{\sigma_u}{|\sigma_u|} + \big(\delta' \cdot \frac{\sigma_v}{|\sigma_v|}\big)\frac{\sigma_v}{|\sigma_v|}$. By Theorem 1.17 (Polar Coordinates), since $|\delta'(s)| = 1$ for all s, we can choose a smooth function $\theta : I \to \mathbb{R}$ (which is unique up to the addition of an integer multiple of 2π) such that $\cos\theta(s) = \delta'(s) \cdot \frac{\sigma_u(\beta(s))}{|\sigma_u(\beta(s))|}$ and $\sin\theta(s) = \delta'(s) \cdot \frac{\sigma_v(\beta(s))}{|\sigma_v(\beta(s))|}$ for all $s \in I$ so that

$$\delta'(s) = \cos\theta(s)\,\frac{\sigma_u(\beta(s))}{|\sigma_u(\beta(s))|} + \sin\theta(s)\,\frac{\sigma_v(\beta(s))}{|\sigma_v(\beta(s))|}$$

The function $\theta(s)$ measures the angle from $\sigma_u(\beta(s))$ counterclockwise to $\delta'(s)$ in the tangent space with ordered orthonormal basis $\big\{\frac{\sigma_u}{|\sigma_u|}, \frac{\sigma_v}{|\sigma_v|}\big\}$.

3.13 Theorem: *(Geodesic Curvature in Orthogonal Coordinates) Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$ for which $g = g(u,v)$ is diagonal for all $(u,v) \in U$. Let $\beta : I \subseteq \mathbb{R} \to U$ be a smooth regular curve which is parametrized by arclength with respect to g, so we have $|\delta'(s)| = 1$ for all $s \in I$ where $\delta(s) = \sigma(\beta(s))$. Let $\theta = \theta(s)$ be a smooth function such that*

$$\delta'(s) = \cos\theta(s)\,\frac{\sigma_u(\beta(s))}{|\sigma_u(\beta(s))|} + \sin\theta(s)\,\frac{\sigma_v(\beta(s))}{|\sigma_v(\beta(s))|}.$$

(1) For $\beta(s) = \big(u(s), v(s)\big)$ we have

$$\cos\theta = \sqrt{g_{11}}\ u'\ ,\ \sin\theta = \sqrt{g_{22}}\ v'.$$

(2) The geodesic curvature $k_g = k_g(s)$ of β at s is related to the geodesic curvatures k_1 and k_2 of the coordinate lines at the point $\beta(s) \in U$ by

$$k_g = \theta' + \cos\theta\, k_1 + \sin\theta\, k_2\,.$$

Proof: To prove Part 1, note that since $\delta' = \cos\theta\frac{\sigma_u}{|\sigma_u|} + \sin\theta\frac{\sigma_v}{|\sigma_v|}$ and $\delta(s) = \sigma(\beta(s))$, we have

$$\begin{aligned}\cos\theta &= \delta' \bullet \tfrac{\sigma_u}{|\sigma_u|} = (\sigma_u u' + \sigma_v v') \bullet \tfrac{\sigma_u}{|\sigma_u|} = \tfrac{\sigma_u \bullet \sigma_u}{|\sigma_u|}\, u' = \sqrt{g_{11}}\, u' \\ \sin\theta &= \delta' \bullet \tfrac{\sigma_v}{|\sigma_v|} = (\sigma_u u' + \sigma_v v') \bullet \tfrac{\sigma_v}{|\sigma_v|} = \tfrac{\sigma_v \bullet \sigma_v}{|\sigma_v|}\, v' = \sqrt{g_{22}}\, v'\end{aligned}$$

Fix $s_0 \in I$ and let $p = (a,b) = \beta(s_0)$. Consider three curves in U through the point p, the given curve $\beta = \beta(s)$ and the two coordinate lines $\beta_1(t) = \big(u_1(t), b\big)$ with $u_1(0) = a$ and $\beta_2(t) = \big(a, v_2(t)\big)$ with $v_2(0) = b$, with all three curves parametrized by arclength with respect to g, so that we have $|\delta'(s)| = 1$ for all s and $|\delta_1(t)| = |\delta_2(t)| = 1$ for all t, where $\delta(s) = \sigma(\beta(s))$, $\delta_1(t) = \sigma(\beta_1(t))$ and $\delta_2(t) = \sigma(\beta_2(t))$. Following each curve, we have orthonormal bases: we have $\{T, M, N\}$ where $T = T(s) = \delta'(s)$, $N = N(s) = n(\beta(s))$ and $M(s) = N(s) \times T(s)$, and we have $\{T_1, M_1, N_1\}$ and $\{T_2, M_2, N_2\}$ where $T_j = T_j(t) = \delta_j{}'(t)$, $N_j(t) = n(\beta_j(t))$ and $M_j(t) = N_j(t) \times T_j(t)$. At p, we have $k_g = k_g(\beta)(s_0) = \delta''(s_0) \bullet M(s_0)$ and for $j = 1, 2$ we have $k_j = k_g(\beta_j)(0) = \delta_j{}''(0) \bullet M_j(0)$.

Since $\delta_1(t) = \sigma(\beta_1(t)) = \sigma(u_1(t), b)$ we have $T_1 = \delta_1{}' = \sigma_u\, u_1{}'$. On the other hand we have $T_1 = \frac{\sigma_u}{|\sigma_u|}$ and so $u_1{}' = \frac{1}{|\sigma_u|}$. Similarly $T_2 = \delta_2{}' = \sigma_v\, v_2{}' = \frac{\sigma_v}{|\sigma_v|}$ so that $v_2{}' = \frac{1}{|\sigma_v|}$. Since $T_1 = \delta_1{}' = \frac{\sigma_u}{|\sigma_u|}$ we have $T_1{}' = \delta_1{}'' = \frac{\partial}{\partial u}\big(\frac{\sigma_u}{|\sigma_u|}\big)\, u_1{}' = \frac{1}{|\sigma_u|}\,\frac{\partial}{\partial u}\big(\frac{\sigma_u}{|\sigma_u|}\big)$. Similarly, we can use the fact that $T_2 = \delta_2{}' = \frac{\sigma_v}{|\sigma_v|}$, $M_1 = \frac{\sigma_v}{|\sigma_v|}$ and $M_2 = -\frac{\sigma_u}{|\sigma_u|}$ to obtain similar formulas for $T_2{}'$, $M_1{}'$ and $M_2{}'$. For all t we have

$$\begin{aligned}T_1{}' = \delta_1{}'' = \tfrac{1}{|\sigma_u|}\,\tfrac{\partial}{\partial u}\big(\tfrac{\sigma_u}{|\sigma_u|}\big) \quad &,\ M_1{}' = \tfrac{1}{|\sigma_u|}\,\tfrac{\partial}{\partial u}\big(\tfrac{\sigma_v}{|\sigma_v|}\big)\,, \\ T_2{}' = \delta_2{}'' = \tfrac{1}{|\sigma_v|}\,\tfrac{\partial}{\partial v}\big(\tfrac{\sigma_v}{|\sigma_v|}\big) \quad &,\ M_2{}' = -\tfrac{1}{|\sigma_v|}\,\tfrac{\partial}{\partial v}\big(\tfrac{\sigma_u}{|\sigma_u|}\big)\,.\end{aligned}$$

Since $\delta(s) = \sigma(\beta(s)) = \sigma(u(s), v(s))$ we have $T = \delta' = \sigma_u\, u' + \sigma_v\, v'$. On the other hand, we have $T = \delta' = \cos\theta\frac{\sigma_u}{|\sigma_u|} + \sin\theta\frac{\sigma_v}{|\sigma_v|}$ and so $u' = \frac{\cos\theta}{|\sigma_u|}$ and $v' = \frac{\sin\theta}{|\sigma_v|}$. Since $\delta' = T = \cos\theta\frac{\sigma_u}{|\sigma_u|} + \sin\theta\frac{\sigma_v}{|\sigma_v|}$ and $M = -\sin\theta\frac{\sigma_u}{|\sigma_u|} + \cos\theta\frac{\sigma_v}{|\sigma_v|}$, for all s we have

$$\begin{aligned}\delta'' = T' &= -\theta'\,\sin\theta\,\tfrac{\sigma_u}{|\sigma_u|} + \cos\theta\,\tfrac{\partial}{\partial u}\big(\tfrac{\sigma_u}{|\sigma_u|}\big)\, u' + \cos\theta\,\tfrac{\partial}{\partial v}\big(\tfrac{\sigma_u}{|\sigma_u|}\big)\, v' \\ &\quad + \theta'\,\cos\theta\,\tfrac{\sigma_v}{|\sigma_v|} + \sin\theta\,\tfrac{\partial}{\partial u}\big(\tfrac{\sigma_v}{|\sigma_v|}\big)\, u' + \sin\theta\,\tfrac{\partial}{\partial v}\big(\tfrac{\sigma_v}{|\sigma_v|}\big)\, v' \\ &= \theta' M + \cos\theta\,\tfrac{\partial}{\partial u}\big(\tfrac{\sigma_u}{|\sigma_u|}\big)\,\tfrac{\cos\theta}{|\sigma_u|} + \cos\theta\,\tfrac{\partial}{\partial v}\big(\tfrac{\sigma_u}{|\sigma_u|}\big)\,\tfrac{\sin\theta}{|\sigma_v|} \\ &\quad + \sin\theta\,\tfrac{\partial}{\partial u}\big(\tfrac{\sigma_v}{|\sigma_v|}\big)\,\tfrac{\cos\theta}{|\sigma_u|} + \sin\theta\,\tfrac{\partial}{\partial v}\big(\tfrac{\sigma_v}{|\sigma_v|}\big)\,\tfrac{\sin\theta}{|\sigma_v|}\end{aligned}$$

In particular, when $s = s_0$ and $t = 0$ we have

$$\delta'' = T' = \theta' M + \cos^2\theta\, T_1' - \cos\theta\sin\theta\, M_2' + \cos\theta\sin\theta\, M_1' + \sin^2\theta\, T_2'\,,$$
$$M = -\sin\theta\, T_1 + \cos\theta\, T_2\,.$$

Since $1 = T_j \bullet T_j$ and $1 = M_j \bullet M_j$, differentiating gives $T_j' \bullet T_j = 0$ and $M_j' \bullet M_j = 0$. At $t{=}0$ we have $M_1 {=} T_2$ and $M_2 {=} {-}T_1$. so $M_1' \bullet T_2 {=} M_1' \bullet M_1 {=} 0$ and $M_1' \bullet T_2 {=} {-}M_1' \bullet M_1 {=} 0$. Taking the dot product of δ'' with M (as above), using these orthogonality formulas gives

$$\delta'' \bullet M = \theta' \;+ \cos^3\theta\; T_1' \bullet T_2 - \cos^2\theta\sin\theta\; M_2' \bullet T_2 - \cos\theta\sin^2\theta\; M_1' \bullet T_1 - \sin^3\theta\; T_2' \bullet T_1.$$

Differentiating $M_j \bullet T_j = 0$ gives $M_j' \bullet T_j + M_j \bullet T_j' = 0$, so that $M_j' \bullet T_j = -T_j' \bullet M_j$, and so $T_1' \bullet T_2 = T_1' \bullet M_1 = k_1$, $M_2' \bullet T_2 = -T_2' \bullet M_2 = -k_2$, $M_1' \bullet T_1 = -T_1' \bullet M_1 = -k_1$ and $T_2' \bullet T_1 = -T_2' \bullet M_2 = -k_2$. Substitute these into the above formula for $\delta'' \bullet M$ to get

$$\begin{aligned} k_g = \delta'' \bullet M &= \theta' + \cos^3\theta\; k_1 + \cos^2\theta\sin\theta\; k_2 + \cos\theta\sin^2\theta\; k_1 + \sin^3\theta\; k_2 \\ &= \theta' + \cos\theta\; k_1 + \sin\theta\; k_2, \end{aligned}$$

as required.

Green's Theorem

3.14 Theorem: *(Green's Theorem) Let U and V be open sets in $\mathbb{R}^2$ such that V contains the triangle $\Delta = \big\{(x,y)\in\mathbb{R}^2 \,\big|\, x\geq 0, y\geq 0, x{+}y\leq 1\big\}$. Let $\phi : U \to V$ be a smooth regular positive change of coordinates with inverse $\psi = \phi^{-1} : V \to U$. Let $R = \psi(\Delta)$ be the image of Δ in U and let $\alpha_1, \alpha_2, \alpha_3 : [0,1] \to U$ be the edges of R given by $\alpha_1(t) = \psi(0, 1{-}t)$, $\alpha_2(t) = \psi(t, 0)$ and $\alpha_3(t) = \psi(1{-}t, t)$. Let $F : U \to \mathbb{R}^2$ be a smooth map given by $F = (P, Q)$ where $P, Q : U \to \mathbb{R}$. Then*

$$\iint_R \Big(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\Big)\, du\, dv = \sum_{j=1}^{3} \int_{t=0}^{1} F\big(\alpha_j(t)\big) \bullet \alpha_j'(t)\, dt.$$

Proof: Let $\delta_1, \delta_2, \delta_3 : [0,1] \to V$ be the edges of Δ given by $\delta_1(t) = (0, 1{-}t)$, $\delta_2(t) = (t, 0)$ and $\delta_3(t) = (1{-}t, t)$ so that $\alpha_j(t) = \psi(\delta_j(t))$, and let $G = (L, M) : V \to \mathbb{R}^2$ be given by $G(x,y) = D\psi(x,y)^T F(\psi(x,y))$, that is

$$\begin{pmatrix} L \\ M \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Then we have $L = Pu_x + Qv_x$ and $M = Pu_y + Qv_y$ so that

$$\begin{aligned} L_y &= (P_u u_y + P_v v_y) u_x + P u_{xy} + (Q_u u_y + Q_v v_y) v_x + Q v_{xy} \\ M_x &= (P_u u_x + P_v v_x) u_y + P u_{yx} + (Q_u u_x + Q_v v_x) v_y + Q v_{yx} \end{aligned}$$

and hence

$$M_x - L_y = (Q_u - P_v)(u_x v_y - u_y v_x) = (Q_u - P_v)\det D\psi.$$

By the change of variables formula for integration, we have

$$\begin{aligned} \iint_R \Big(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\Big)\, du\, dv &= \iint_\Delta \Big(\frac{\partial Q}{\partial u}\big(\psi(x,y)\big) - \frac{\partial P}{\partial v}\big(\psi(x,y)\big)\Big)\big(\det D\psi\big)\, dx\, dy \\ &= \iint_\Delta \Big(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\Big)\, dx\, dy. \end{aligned}$$

Also, when $\delta:[0,1]\to V$ and $\alpha(t)=\psi(\delta(t))$ we have $\alpha'(t)=D\psi\big(\delta(t)\big)\delta'(t)$ so that

$$\int_{t=0}^{1} F\big(\alpha(t)\big)\cdot\alpha'(t)\,dt=\int_0^1 \alpha'(t)^T F\big(\alpha(t)\big)\,dt=\int_0^1 \delta'(t)^T D\psi\big(\delta(t)\big)^T F\big(\psi(\delta(t))\big)\,dt$$
$$=\int_0^1 \delta'(t)G\big(\delta(t)\big)\,dt=\int_0^1 G\big(\delta(t)\big)\cdot\delta'(t)\,dt.$$

Thus it suffices to show that

$$\iint_\Delta\Big(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\Big)dx\,dy=\sum_{j=1}^{3}\int_0^1 G\big(\delta_j(t)\big)\cdot\delta_j'(t)\,dt.$$

And indeed, we have

$$\int_0^1 G\big(\delta_1(t)\big)\cdot\delta_1'(t)\,dt=\int_0^1\big(L(0,1-t),M(0,1-t)\big)\cdot(0,-1)\,dt=-\int_0^1 M(0,1-t)\,dt$$
$$\int_0^1 G\big(\delta_2(t)\big)\cdot\delta_2'(t)\,dt=\int_0^1\big(L(t,0),M(t,0)\big)\cdot(1,0)\,dt=\int_0^1 L(t,0)\,dt$$
$$\int_0^1 G\big(\delta_3(t)\big)\cdot\delta_3'(t)\,dt=\int_0^1\big(L(1-t,t),M(1-t,t)\big)\cdot(-1,1)\,dt$$
$$=\int_0^1 M(1-t,t)\,dt-\int_0^1 L(1-t,t)\,dt,$$

and

$$\iint_\Delta\Big(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\Big)\,dx\,dy=\int_{y=0}^1\Big(\int_{x=0}^{1-y}\frac{\partial M}{\partial x}\,dx\Big)\,dy-\int_{x=0}^1\Big(\int_{y=0}^{1-x}\frac{\partial L}{\partial y}\,dy\Big)\,dx$$
$$=\int_{y=0}^1\Big[M(x,y)\Big]_{x=0}^{1-y}dy-\int_{x=0}^1\Big[L(x,y)\Big]_{y=0}^{1-x}dx$$
$$=\int_{y=0}^1 M(1-y,y)\,dy-\int_{y=0}^1 M(0,y)\,dy-\int_{x=0}^1 L(x,1-x)\,dx+\int_{x=0}^1 L(x,0)\,dx$$
$$=\int_{t=0}^1 M(1-t,t)\,dt-\int_{t=0}^1 M(0,1-t)\,dt-\int_{t=0}^1 L(1-t,t)\,dt+\int_{t=0}^1 L(t,0)\,dt.$$

The Gauss-Bonnet Formula

3.15 Definition: Let $\sigma:U\subseteq\mathbb{R}^2\to\mathbb{R}^3$ be a smooth regular surface, let $R\subseteq U$ be a Jordan region, let $\alpha:[a,b]\subseteq\mathbb{R}\to U$ be a smooth regular curve, and let $\gamma(t)=\sigma(\alpha(t))$. The **integral of the geodesic curvature** of α on $[a,b]$ is given by

$$\int_\alpha k_g\,dL=\int_a^b k_g(\alpha)(t)\big|\gamma'(t)\big|\,dt=\int_a^b k_g(\alpha)(t)\sqrt{\alpha'(t)^T g(\alpha(t))\alpha'(t)}\,dt$$

and the **integral of the Gaussian curvature** of σ on R is given by

$$\iint K\,dA=\iint K_\sigma\,dA=\iint_R K_\sigma(u,v)\sqrt{\det g_\sigma(u,v)}\,du\,dv.$$

3.16 Theorem: *(Change of Coordinates) Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface, let $R \subseteq U$ be a Jordan region, let $\alpha : [a,b] \subseteq \mathbb{R} \to U$ be a smooth regular curve, and let $\gamma(t) = \sigma(\alpha(t))$.*

(1) Let $r : [a,b] \to [c,d]$ is a regular change of parametrization and let $\lambda(r) = \alpha(t(r))$. Then

$$k_g(\lambda)(r(t)) = \pm\, k_g(\alpha)(t) \quad \text{and} \quad \int_\lambda k_g\, dL = \pm \int_\alpha k_g\, dL$$

where we use the $+$ sign when $r'(t) > 0$ for all t and the $-$ sign when $r'(t) < 0$ for all t.

(2) Let $\phi : U \to V$ be a smooth regular change of coordinates with inverse $\psi : V \to U$, let $\rho : V \to \mathbb{R}^3$ be given by $\rho(x,y) = \sigma(\psi(x,y))$ and let $\lambda : [a,b] \to V$ be given by $\lambda(t) = \phi(\alpha(t))$. Then

$$k_g(\lambda)(t) = \pm\, k_g(\alpha)(t)\ ,\ \int_\lambda k_g\, dL = \pm \int_\alpha k_g dL \quad \text{and} \quad \iint_R K_\sigma\, dA = \iint_{\phi(R)} K_\rho\, dA.$$

where we use the $+$ sign when ϕ preserves orientaton (that is when $\det D\phi(u,v) > 0$ for all u,v) and the $-$ sign when ϕ reverses orientation.

Proof: We prove part 1, in the case that $r = r(t)$ reverses direction, and leave the rest of the proof as an exercise. Suppose that $r : [a,b] \to [c,d]$ reverses direction, so $r'(t) < 0$ for all t and $r(a) = d$ and $r(b) = c$. We have $\alpha : [a,b] \to U$. To define $k_g(\alpha)$ we let $\gamma(t) = \sigma(\alpha(t))$ and we reparametrize by arclength to get $\beta(s(t)) = \alpha(t)$ and $\delta(s(t)) = \gamma(t)$ with $s(t) = \int_a^t |\gamma'(x)|\, dx$. We also have $\lambda : [c,d] \to U$ given by $\lambda(r(t)) = \alpha(t)$. To define $k_g(\lambda)$ we let $\nu(r) = \sigma(\lambda(r))$ and reparametrize by arclength to get $\mu(q(r)) = \lambda(r)$ and $\xi(q(r)) = \nu(r)$ with $q = q(r) = \int_c^r |\nu'(y)|\, dy$. Note that $\gamma(t) = \sigma(\alpha(t)) = \sigma(\lambda(r(t))) = \nu(r(t))$, so

$$\tfrac{d}{dt} q(r(t))) = q'(r(t)) r'(t) = \big|\nu'(r(t))\big| r'(t) = -\big|\nu'(r(t)) r'(t)\big| = -\big|\gamma'(t)\big| = -s'(t).$$

Since

$$\xi(q(r(t))) = \nu(r(t)) = \gamma(t) = \delta(s(t)),$$

we have $\xi'(q(r(t))) \frac{d}{dt} q(r(t)) = \delta'(s(t))\, s'(t)$ hence $\xi'(q(r(t))) = -\delta'(s(t)$, and similarly $\xi''(q(r(t))) = +\delta''(s(t))$. Thus

$$\begin{aligned} k_g(\alpha)(t) &= k_g(\beta)(s(t)) = \delta''(s(t)) \cdot \Big(\tfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} (\beta(s(t)) \times \delta'(s(t)) \Big) \\ &= -\xi''(q(r(t))) \cdot \Big(\tfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} (\mu(q(r(t)))) \times \xi'(q(r(t))) \Big) = -k_g(\mu)(q(r(t)))) = -k_g(\lambda)(r(t)). \end{aligned}$$

Thus

$$\begin{aligned} \int_\alpha k_g dL &= \int_{t=a}^b k_g(\alpha) |\gamma'(t)|\, dt = \int_{t=a}^b -k_g(\lambda)(r(t)) \big|\nu'(r(t)) r'(t)\big|\, dt \\ &= \int_{t=a}^v k_g(\lambda)(r(t)) \big|\nu'(r(t))\big|\, r'(t)\, dt = \int_d^c k_g(\lambda)(r) |\nu'(r)|\, dr = -\int_\lambda k_g dL. \end{aligned}$$

3.17 Theorem: *(The Gauss-Bonnet Formula in Orthogonal Coordinates) Let $U \subseteq \mathbb{R}^2$ and let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$ for which $g(u,v)$ is diagonal for all $(u,v) \in U$. Let $\phi : U \to V$ be a smooth regular positive change of coordinates, with inverse $\psi = \phi^{-1} : V \to U$, such that V contains the triangle $\Delta = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, x \geq 0, y \geq 0, x+y \leq 1\big\}$. Let $R = \psi(\Delta)$ be the image of Δ in U, and let α_1, α_2 and α_3 be the edges of R given by $\alpha_1 = \psi(0, 1-t)$, $\alpha_2 = \psi(t, 0)$ and $\alpha_3(t) = (1-t, t)$ for $0 \leq t \leq 1$. Let $\gamma_j(t) = \sigma(\alpha_j(t))$, choose smooth functions $\theta_j(t)$ so that $\frac{\gamma_j'(t)}{|\gamma_j'(t)|} = \cos\theta(t)\, \frac{\sigma_u}{|\sigma_u|}(\gamma_j(t)) + \sin\theta(t)\, \frac{\sigma_v}{|\sigma_v|}(\gamma_j(t))$, and let $\Delta_j\theta = \theta_j(1) - \theta_j(0)$ for $j \in \{1,2,3\}$. Then*

$$\iint_R K\, dA + \sum_{j=1}^{3} \int_{\alpha_j} k_g\, dL = \sum_{j=1}^{3} \Delta\theta_j\,.$$

Proof: Fix $j \in \{1,2,3,\}$ and reparametrize by arclength using $s : [0,1] \to [0,\ell]$ given by $s(t) = \int_0^t |\gamma_j(r)|\, dr$ and $\alpha_j(t) = \beta_j(s(t))$, $\gamma_j(t) = \delta_j(s(t))$. Since $\delta_j'(s) = \frac{\gamma_j{}'(t(s))}{|\gamma_j{}'(t(s))|}$ we have

$$\delta_j'(s) = \cos\Theta_j(s)\, \tfrac{\sigma_u}{|\sigma_u|}(\delta_j(s)) + \sin\Theta_j(s)\, \tfrac{\sigma_v}{|\sigma_v|}(\delta_j(s))$$

where $\Theta_j(s(t)) = \theta_j(t)$. By the change of coordinates formula (Part 1 of Theorem 3.16) and the formula for $k_g(\beta)$ in orthogonal coordinates (Part 2 of Theorem 3.13), we have

$$\begin{aligned} \int_{\alpha_j} k_g\, dL = \int_{\beta_j} k_g\, dL = \int_{s=0}^{\ell} k_g(\beta_j)(s)|\delta'(s)|\, ds = \int_{s=0}^{\ell} k_g(\beta)(s)\, ds \\ = \int_{s=0}^{\ell} \Theta_j'(s) + k_1(s)\cos\Theta_j(s) + k_2(s)\sin\Theta_j(s), \end{aligned}$$

where k_1 and k_2 are the geodesic curvatures of the coordinate lines. Since $\Theta_j(s(t)) = \theta_j(t)$ we have

$$\int_{s=0}^{\ell} \Theta_j'(s)\, ds = \int_{t=0}^{1} \Theta_j'(s(t))s'(t)\, dt = \int_{t=0}^{1} \theta_j'(t)\, dt = \Delta\theta_j\,.$$

By Theorem 3.11 and Part 1 of Theorem 3.13, we have $k_1 = -\frac{(g_{11})_v}{2g_{11}\sqrt{g_{22}}}$, $k_2 = \frac{(g_{22})_u}{2g_{22}\sqrt{g_{11}}}$, $\cos\Theta_j = \sqrt{g_{11}}\; u'$ and $\sin\Theta_j = \sqrt{g_{22}}\; v'$ where $\beta_j(s) = \big(u(s), v(s)\big)$, and so

$$\begin{aligned} &\int_{s=0}^{\ell} k_1(s)\cos\Theta_j(s) + k_2(s)\sin\Theta_j(s)\; ds \\ &\quad = \int_{s=0}^{\ell} -\tfrac{(g_{11})_v}{2g_{11}\sqrt{g_{22}}} \cdot \sqrt{g_{11}}\; u' + \tfrac{(g_{22})_u}{2g_{22}\sqrt{g_{11}}} \cdot \sqrt{g_{22}}\; v'\; ds \\ &\quad = \int_{s=0}^{\ell} F\big(\beta_j(s)\big) \bullet \beta_j'(s)\, ds = \int_{t=0}^{1} F\big(\alpha_j(t)\big) \bullet \alpha_j'(t)\; dt \end{aligned}$$

where $F = (P,Q)$ with $P = -\frac{(g_{11})_v}{2\sqrt{g_{11}g_{22}}}$ and $Q = \frac{(g_{22})_u}{2\sqrt{g_{11}g_{22}}}$. Using Green's Theorem, then the formula for K in orthogonal coordinates (Theorem 3.10), we have

$$\begin{aligned} \sum_{j=1}^{3} \int_{t=0}^{1} F\big(\alpha_j(t)\big) \bullet \alpha_j'(t)\, dt &= \iint_R \frac{\partial}{\partial u}\, \frac{(g_{22})_u}{2\sqrt{g_{11}g_{22}}} + \frac{\partial}{\partial v}\, \frac{(g_{11})_v}{2\sqrt{g_{11}}g_{22}}\; du\, dv \\ &= \iint_R -\sqrt{g_{11}g_{22}}\; K\; du\, dv = -\iint_R K\, \sqrt{\det g}\; du\, dv = -\iint_R K\; dA \end{aligned}$$

so that $\displaystyle\sum_{j=1}^{3} \int_{\alpha_j} k_g\, dL = \sum_{j=1}^{3} \Delta\theta_j + \sum_{j=1}^{3} \int_{t=0}^{1} F(\alpha_j(t)) \bullet \alpha_j'(t)\, dt = \sum_{j=1}^{3} \Delta\theta_j - \iint_R K\, dA.$

3.18 Note: In the Gauss-Bonnet formula, if we let $\epsilon_1, \epsilon_2, \epsilon_3 \in [0, \pi]$ be the external angles at the vertices of the curved image triangle $\sigma(R)$ $\big($that is the angles from from ${\gamma_1}'(1)$ to ${\gamma_2}'(0)$, from $\gamma_2'(1)$ to $\gamma_3'(0)$, and from $\gamma_3'(1)$ to $\gamma_1'(0)\big)$ then it is intuitively apparent (but surprisingly difficult to prove) that

$$\sum_{j=1}^{3} \Delta\theta_j + \sum_{j=1}^{3} \epsilon_j = 2\pi.$$

Using this formula, then we can write the conclusion of the Gauss-Bonnet Formula as

$$\iint_R K\, dA + \sum_{j=1}^{3} \int_{\alpha_j} k_g\, dL + \sum_{j=1}^{3} \epsilon_j = 2\pi\,.$$

Let us describe the ideas which can be used to prove that $\sum_{j=1}^{3} \Delta\theta_j + \sum_{j=1}^{3} \epsilon_j = 2\pi$. It is not hard to argue that the sum on the left must be of the form $2\pi\, k$ for some $k \in \mathbb{Z}$, the hard part is to show that $k = 1$. If the composite map $\sigma(\psi(x, y))$ was an affine map (that is a linear map composed with a translation) then it would send the triangle Δ to a triangle in $\mathbb{R}^3$, and the edges $\gamma_j(t)$ would be straight lines so that $\gamma_j'(t) = 0$ hence $\Delta\theta_j = 0$, and the internal angles would add up to π so that the external angles would add to 2π. In general, the map $\sigma \circ \psi$ can be approximated by its linearization (which is an affine map). We can partition the edges of Δ into n equal sized subintervals, and cut the triangle into n^2 small congruent triangles, such that in each triangle the map $\sigma \circ \psi$ is approximated by its linearization sufficiently closely that the sum $\sum \Delta\theta_j + \sum \epsilon_j$ is forced to be close (say within π) to the corresponding sum for the affine map, which is exactly equal to 2π. Since the sum is an integer multiple of 2π and it is close to 2π (to within π) it must be exactly equal to 2π, so the formula we are trying to prove holds for each of the n^2 small triangles. To prove that it holds for the original triangle Δ, add all of the terms in all of the sums for the n^2 small triangles to get

$$\sum_{i=1}^{n^2} \Big(\sum_{j=1}^{3} \Delta\theta_{i,j} + \sum_{j=1}^{3} \epsilon_{i,j} \Big) = n^2 \cdot 2\pi.$$

When two small triangles meet along an edge inside Δ, the corresponding terms $\Delta\theta_{i,j}$ cancel, and the sum of all the remaining terms $\Delta\theta_{i,j}$ is equal to the sum $\sum_{j=1}^{3} \Delta\theta_j$. When Δ is cut into n^2 small triangles, there are a total of $\frac{(n+2)(n+1)}{2}$ vertices: there are 3 corner vertices, there are $3(n-1)$ vertices along the edges of Δ, and there are $\frac{(n-1)(n-2)}{2}$ vertices inside Δ. At each of the $3(n-1)$ edge vertices, three small triangles meet, the three internal angles meeting at the vertex add up to π, and the three external angles add up to $3\pi - \pi = 2\pi$. At each of the $\frac{(n-1)(n-2)}{2}$ internal vertices, six small triangles meet, the six internal angles meeting at the vertex add up to 2π, and the six external angles add up to $6\pi - 2\pi = 4\pi$. It follows that the sum of all the terms $\epsilon_{i,j}$ is equal to

$$\sum_{i,j} \epsilon_{i,j} = \sum_{j=1}^{3} \epsilon_j + 3(n-1) \cdot 2\pi + \tfrac{(n-2)(n-1)}{2} \cdot 4\pi = \sum_{j=1}^{3} \epsilon_j + (n^2 - 1) \cdot 2\pi.$$

Thus we have $n^2 \cdot 2\pi = \sum_{i,j} \Delta\theta_{i,j} + \sum_{i,j} \epsilon_{i,j} = \sum_{j=1}^{3} \Delta\theta_j + \sum_{j=1}^{3} \epsilon_j + (n^2 - 1) \cdot 2\pi$ and hence

$$\sum_{j=1}^{3} \Delta\theta_j + \sum_{j=1}^{3} \epsilon_j = 2\pi.$$

3.19 Note: In the Gauss-Bonnet formula, with the conclusion written in the form

$$\iint_R K\,dA + \sum_{j=1}^{3}\int_{\alpha_j} k_g\,dL + \sum_{j=1}^{3}\epsilon_j = 2\pi\,,$$

it is not necessary to assume that we are using orthogonal coordinates, that is we do not need to assume that $g(u,v)$ is diagonal for all $(u,v)\in U$. Let us sketch the proof. By Theorem 3.7, we know that each point $p\in U$ is contained in an open set U_p in which we can change to orthogonal coordinates. As in the above note, we can partition the edges of Δ into n equal subintervals, and cut the triangle into n^2 small congruent triangles, such that the image of each of the small triangles is contained in one of the sets U_p in which we can change to orthogonal coordinates. By switching the order of the two variables, if necessary, we may assume that all of the change of coordinate maps are orientation-preserving. Using a change of coordinates on the i^{th} triangle does not change its external angles $\epsilon_{i,j}$, and (by Theorem 3.16) it does not change the values of the integrals, so the Gauss-Bonnet formula holds for each of the n^2 small triangles. To prove that it holds for the original triangle Δ, we can add the contributions of all the terms in all of the sums for the n^2 small triangles, as we did in the previous note. We now restate the Gauss-Bonnet Formula in light of Notes 3.18 and 3.19.

3.20 Theorem: *(The Gauss-Bonnet Formula) Let $\Delta=\big\{(u,v)\,\big|\,u\geq 0, v\geq 0, u+v\leq 1\big\}$, and let $\alpha_1,\alpha_2,\alpha_3:[0,1]\to\Delta$ be given by $\alpha_1(t)=(0,1-t)$, $\alpha_2(t)=(t,0)$ and $\alpha_3(t)=(1-t,t)$. Let $U\subseteq\mathbb{R}^2$ be open with $\Delta\subseteq U$ and let $\sigma:U\subseteq\mathbb{R}^2\to\mathbb{R}^3$ be a smooth regular surface. Let $\epsilon_1,\epsilon_2,\epsilon_3$ be the external angles of $\sigma(\Delta)$ at the vertices. Then*

$$\iint_\Delta K_\sigma\,dA + \sum_{j=1}^{3}\int_{\alpha_j} k_g\,dL + \sum_{j=1}^{3}\epsilon_j = 2\pi.$$

3.21 Remark: A smooth regular local parametrized surface in $\mathbb{R}^n$ is a smooth regular map $\sigma:U\subseteq\mathbb{R}^2\to\mathbb{R}^n$. Informally, it is more natural to think of the image $S=\sigma(U)$ as a surface rather than to think of the map σ as the surface. From the point of view of writing down formulas for geometric properties, such as Gaussian curvature, it is more convenient to work with the map σ. We now wish to define a surface in $\mathbb{R}^n$ as a subset of $\mathbb{R}^n$. Roughly speaking, we shall define a smooth global surface in $\mathbb{R}^n$ to be a set which is covered by the images of some injective smooth regular parametrized surfaces $\sigma:U\subseteq\mathbb{R}^2\to\mathbb{R}^n$. As an example, note that the unit sphere in $\mathbb{R}^3$ is not equal to the image of a single injective regular smooth parametrized surface, but it is covered by eight open hemispheres, each of which is equal to such an image. We shall then give a global version of the Gauss-Bonnet Formula which applies to global surfaces.

3.22 Definition: Recall that a **homeomorphism** is a bijective function such that both the function and its inverse are continuous. A **smooth regular global surface** in $\mathbb{R}^n$ (or a **smooth regular 2-dimensional submaifold** of $\mathbb{R}^n$) is a set $S\subseteq\mathbb{R}^n$ for which there exists a set $\mathcal{A}$ of smooth regular homeomorphisms $\sigma:U_\sigma\subseteq\mathbb{R}^2\to S\cap W_\sigma\subseteq\mathbb{R}^n$, where $U_\sigma\subseteq\mathbb{R}^2$ and $W_\sigma\subseteq\mathbb{R}^n$ are open sets, such that

(1) the images of the maps σ cover S, that is $S=\bigcup_{\sigma\in\mathcal{A}}\sigma(U_\sigma)=\bigcup_{\sigma\in\mathcal{A}}(S\cap W_\sigma)$, and
(2) when the image of two maps $\sigma,\rho\in\mathcal{A}$ intersect, the maps $\rho^{-1}\sigma$ and $\sigma^{-1}\rho$ are smooth (so the map $\rho^{-1}\sigma$ is a smooth regular change of coordinates from the set $\sigma^{-1}(S\cap W_\sigma\cap W_\rho)$ to the set $\rho^{-1}(S\cap W_\sigma\cap W_\rho)$).

The set $\mathcal{A}$ is called an **atlas** for S, and the maps σ are called (coordinate) **charts** on S.

3.23 Definition: Let Δ be the closed triangle $\Delta = \{(u,v)\in\mathbb{R}^2 \,|\, u,v\geq 0, u^2+v^2\leq 1\}$ and let Δ^o be the open triangle $\Delta^o = \{(u,v)\in\mathbb{R}^2 \,|\, u,v>0, u^2+v^2<1\}$. Let $e_0=(0,0)$, $e_1=(1,0)$ and $e_2=(0,1)$. Let $\alpha_0,\alpha_1,\alpha_2:[0,1]\to\Delta\subseteq\mathbb{R}^2$ be given by $\alpha_0(t)=(1-t,t)$, $\alpha_1(t)=(0,1-t)$ and $\alpha_2(t)=(t,0)$. A **smooth regular triangulated surface** in $\mathbb{R}^3$ is a smooth regular global surface $S\subseteq\mathbb{R}^3$ together with a finite atlas $\mathcal{A}=\{\sigma_1,\cdots,\sigma_n\}$ of smooth regular homeomorphisms $\sigma_i:U_i\subseteq\mathbb{R}^2\to S\cap W_i\subseteq\mathbb{R}^3$, where $U_i\subseteq\mathbb{R}^2$ and $W_i\subseteq\mathbb{R}^3$ are open with $\Delta\subseteq U_i$, such that the following hold:

(1) The images $\sigma_i(\Delta)$ cover S, that is $S=\bigcup_{i=1}^n\sigma_i(\Delta)$.
(2) The images of $\sigma_i(\Delta^o)$ are disjoint, that is $\sigma_i(\Delta^o)\cap\sigma_j(\Delta^o)=\emptyset$ when $i\neq j$.
(3) The edges are joined in pairs: for all indices $i\in\{1,\cdots,n\}$, $j\in\{0,1,2\}$, there exist unique indices $k\in\{1,\cdots,n\}$, $\ell\in\{0,1,2\}$ with $(k,\ell)\neq(i,j)$ such that $\sigma_i(\alpha_j(0,1))\cap\sigma_k(\alpha_\ell(0,1))\neq\emptyset$, and then either $\sigma_i^{-1}\sigma_k$ preserves orientaion in which case $\sigma_i(\alpha_j(t))=\sigma_k(\alpha_\ell(1-t))$ for all $t\in[0,1]$, or $\sigma_i^{-1}\sigma_k$ reverses orientation in which case $\sigma_i(\alpha_j(t))=\sigma_k(\alpha_\ell(t))$ for all $t\in[0,1]$.
(4) Images of vertices are disjoint unless they are are corresponding vertices of a joined pair of edges: if there exist pairs of indices $(i,j)\neq(k,\ell)$ such that $\sigma_i(e_j)=\sigma_k(e_\ell)$ then there exist pairs of indices $(i,j)\neq(k,\ell)$ such that $\sigma_i(\alpha_j(t))=\sigma_k(\alpha_\ell(1-t))$ for all $t\in[0,1]$.

The set $\mathcal{A}=\{\sigma_1,\sigma_2,\cdots,\sigma_n\}$ is called a **triangulation** of S. The **faces** of the triangulation are the images $\sigma_i(\Delta)$ (there are n faces), the **edges** of the triangulation are the images $\sigma_i(\alpha_i[0,1])$ (there are $\frac{3n}{2}$ edges) and the **vertices** of the triangulation are the images $\sigma_i(e_j)$ (the number of vertices depends on how many triangles intersect at each vertex). We shall assume that for each vertex v, the sum of the interior angles at v of all the faces $\sigma_i(\Delta)$ which meet at v is equal to 2π. The **Euler characteristic** of the triangulation is

$$\chi = V - E + F$$

where V is the number of vertices, E is the number of edges, and F is the number of faces.

3.24 Theorem: *(The Gauss-Bonnet Theorem) For a smooth regular triangulated surface S in $\mathbb{R}^3$ with triangulation $\{\sigma_1,\cdots,\sigma_n\}$, we have*

$$\sum_{i=1}^n\iint_\Delta K_{\sigma_i}\,dA = 2\pi\,\chi.$$

Proof: By the Gauss-Bonnet Formula (Theorem 3.20), for each chart σ_i we have

$$\iint_\Delta K_{\sigma_i}\,dA+\sum_{j=0}^2\int_{\alpha_{i,j}}k_g\,dL+\sum_{j=1}^2\epsilon_{i,j}=2\pi$$

where we are writing $\alpha_{i,j}$ to denote the curve α_j on the surface σ_i and $\epsilon_{i,j}$ to denote the external angle of $\sigma_i(\Delta)$ at $\sigma_i(e_j)$. Because the edges are joined together in pairs, when we take the sum over all the charts the terms involving the integrals of the geodesic curvature cancel in pairs (by Theorem 3.16) leaving

$$\sum_{i=1}^n\iint_\Delta K_{\sigma_i}\,dA+\sum_{i,j}\epsilon_{i,j}=2\pi\,n.$$

Let $\varphi_{i,j}$ be the internal angle of $\sigma_i(\Delta)$ at $\sigma_i(e_j)$, that is let $\varphi_{i,j}=\pi-\epsilon_{i,j}$. Since the number of faces is $F=n$ we have $2\pi n=2\pi F$. Since the edges are joined in pairs so the number of edges is $E=\frac{3F}{2}=\frac{3n}{2}$, we have $\sum\limits_{i,j}\pi=3n\pi=2\pi E$. Since there are V vertices and the sum of the internal angles at each vertex is 2π, we have $\sum\limits_{i,j}\varphi_{i,j}=2\pi V$. Thus

$$\sum_{i=1}^n\iint_\Delta K_{\sigma_i}\,dA=2\pi n-\sum_{i,j}\epsilon_{i,j}=2\pi n-\sum_{i,j}\pi+\sum_{i,j}\varphi_{i,j}=2\pi F-2\pi E+2\pi V=2\pi\,\chi.$$

3.25 Remark: When S is a regular global surface in $\mathbb{R}^3$, and $p \in S$ lies in the image of two different coordinate charts σ and ρ, it is easy to check (from our change of coordinate formulas for g and h) that when $\sigma(u,v) = \rho(s,t) = p$ we have $K_\sigma(u,v) = K_\rho(s,t)$. Thus it makes sense to define the Gaussian curvature of S at p to be given by $K_S(p) = K_\sigma(u,v)$ (it would not make sense to do this if the charts were not injective).

We mention several facts, each of which is quite difficult to prove. One fact is that every smooth regular global surface in $\mathbb{R}^3$ can be triangulated. A second fact is that the sum of the integrals of the Gaussian curvatures, which appears in the Gauss-Bonnet Theorem, is independent of the triangularization, so for a smooth regular global surface $S \subseteq \mathbb{R}^3$ we can define the integral of the Gaussian curvature on S to be

$$\iint_S K_S \, dA = \sum_{i=1}^n \iint_\Delta K_{\sigma_i} \, dA$$

where $\{\sigma_1, \cdots, \sigma_n\}$ is any triangularization of S. A third fact (which both follows from and implies the second fact, by the Gauss-Bonnet Theorem) is that every triangulation of a smooth regular global surface S in $\mathbb{R}^3$ has the same Euler characteristic, so we can define the Euler characteristic $\chi(S)$ of the surface S to be equal to the Euler characteristic of any triangulation of S.