Appendix 1. Continued Fractions

1.1 Definition: Let $a_0, a_1, a_2, \cdots \in \mathbf{R}$ with $a_k > 0$. For $n \geq 0$ we write

$$[a_0, a_1, a_2, \cdots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

and

$$[a_0, a_1, a_2, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} = \lim_{n\to\infty} [a_0, a_1, a_2, \cdots, a_n].$$

A **finite continued fraction** is a rational number of the form $[a_0, a_1, \cdots, a_n]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $1 \leq k \leq n$, and an **infinite continued fraction** is a real number of the form $[a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$.

1.2 Theorem: *Every rational number is equal to a finite continued fraction.*

Proof: Let $x = \frac{a}{b}$ with $a \in \mathbf{Z}$ and $b \in \mathbf{Z}^+$. Use the Division Algorithm repeatedly to get

$$a = q_1 b + r_1 \ , \ b = q_2 r_1 + r_2 \ , \ r_1 = q_3 r_3 + r_4 \ , \ \cdots \ , \ r_{n-2} = q_n r_{n-1} + r_n$$

with $0 = r_n < r_{n-1} < \cdots < r_2 < r_1 < b$. Then we have

$$x = \frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{b/r_1} = q_1 + \frac{1}{q_2 + \frac{r_2}{r_1}} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \frac{r_3}{r_2}}} = \cdots = [q_1, q_2, \cdots, q_n].$$

1.3 Remark: Note that when we write a rational number x as a continued fraction $x = [a_0, a_1, \cdots, a_n]$, the integers a_k are not unique because we have

$$[a_0, a_1, \cdots, a_n, 1] = [a_0, a_1, \cdots, a_{n-1}, a_n + 1].$$

1.4 Theorem: *Let $a_0 \in \mathbf{R}$ and let $0 < a_k \in \mathbf{R}$ for $k \geq 1$. For each $n \geq 0$ let $c_n = [a_0, a_1, \cdots, a_n]$. Define sequences $\{p_n\}$ and $\{q_n\}$ recursively by $p_0 = a_0$, $p_1 = a_1 a_0 + 1$ and $p_k = a_k p_{k-1} + p_{k-2}$ for $k \geq 2$, and $q_0 = 1$, $q_1 = a_1$ and $q_k = a_k q_{k-1} + q_{k-2}$ for $k \geq 2$. Then for all $n \geq 0$ we have $c_n = \frac{p_n}{q_n}$.*

Proof: We have $c_0 = [a_0] = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}$ and $c_1 = [a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}$. Let $k \geq 1$ and suppose, inductively, that for $a_0', a_1', \cdots, a_k' \in \mathbf{R}$ with $a_i' > 0$ for $1 \leq i \leq k$ we have $[a_0', a_1', \cdots, a_k'] = \frac{p_k'}{q_k'}$ where $\{p_n'\}$ and $\{q_n'\}$ satisfy the same recursion formulas as $\{p_n\}$ and $\{q_n\}$. Then using $a_i' = a_i$ for $i < k$ and $a_k' = a_k + \frac{1}{a_{k+1}}$, and noting that $p_i' = p_i$ and $q_i' = q_i$ for $i < k$, we have

$$\begin{aligned} c_{k+1} = [a_0, a_1, \cdots, a_{k+1}] &= \left[a_0, a_1, \cdots, a_{k-1}, a_k + \tfrac{1}{a_{k+1}}\right] = \frac{p_k'}{q_k'} = \frac{a_k' p_{k-1}' + p_{k-2}'}{a_k' q_{k-1}' + q_{k-2}'} \\ &= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}} = \frac{a_{k+1} a_k p_{k-1} + p_{k-1} + a_{k+1} p_{k-2}}{a_{k+1} a_k q_{k-1} + q_{k-1} + a_{k+1} q_{k-2}} \\ &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} = \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}. \end{aligned}$$

1.5 Theorem: *Let $a_0 \in \mathbf{Z}$ and let $a_k \in \mathbf{Z}^+$ for $k \geq 1$. Let $c_n = [a_0, a_1, \cdots, a_n]$ for $n \geq 0$. Let $\{p_n\}$ and $\{q_n\}$ be as in Theorem 1.4 so that $c_n = \frac{p_n}{q_n}$. Then*

(1) for all $k \geq 0$ we have $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$,
(2) for all $k \geq 0$ we have $\gcd(p_k, q_k) = 1$,
(3) for all $k \geq 0$ we have $c_{k+1} - c_k = \frac{(-1)^k}{q_{k+1}q_k}$,
(4) the sequence $\{c_n\}$ converges, and
(5) if we let $x = [a_0, a_1, a_2, \cdots] = \lim\limits_{n\to\infty} c_n$ then we have $c_{2k} < x < c_{2k+1}$ for all $k \geq 0$.

Proof: To prove Part (1), note that $p_1q_0 - q_1p_0 = (a_1a_0 + 1)(1) - (a_1)(a_0) = 1$ and that for $k \geq 1$

$$p_{k+1}q_k - q_{k+1}p_k = (a_kp_k + p_{k-1})q_k - (a_kq_k + q_{k-1})p_k = -(p_kq_{k-1} - q_kp_{k-1}).$$

Part (2) follows immediately from Part (1), and Part (3) also follows from Part (1) because

$$c_{k+1} - c_k = \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{p_{k+1}q_k - q_{k+1}p_k}{q_{k+1}q_k} = \frac{(-1)^k}{q_{k+1}q_k}.$$

Since $c_0 = a_0$ and $c_{k+1} - c_k = \frac{(-1)^k}{q_{k+1}q_k}$, we have $c_n = a_0 + \sum\limits_{k=0}^{n-1} \frac{(-1)^k}{q_{k+1}q_k}$ so Parts (4) and (5) both follow from Part (2) by the Alternating Series Test.

1.6 Definition: Let $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$. Then $c_n = [a_0, a_1, \cdots, a_n]$ is called the n^{th} **convergent** of $x = [a_0, a_1, a_2, \cdots]$ and p_n and q_n are called the **numerator** and **denominator** of c_n. Note that $\gcd(p_k, q_k) = 1$ by Part (1) of the above theorem.

1.7 Theorem: *Let $x \in \mathbf{R}$. Then x is irrational if and only if $x = [a_0, a_1, a_2, \cdots]$ for some $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$. In this case we have $a_n = \lfloor x_n \rfloor$ where $\{x_n\}$ is given by*

$$x_0 = x \text{ and } x_{k+1} = \tfrac{1}{x_k - \lfloor x_k \rfloor} \text{ for } k \geq 1.$$

Proof: First let us show that if $x = [a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$ then we must have $x \in \mathbf{R} \setminus \mathbf{Q}$. Let $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$ and let $x = [a_0, a_1, a_2, \cdots]$. For each $k \geq 0$, let $c_k = [a_0, a_1, \cdots, a_n] = \frac{p_k}{q_k}$. Suppose, for a contradiction, that $x = \frac{r}{s}$ with $r \in \mathbf{Z}$ and $s \in \mathbf{Z}^+$. For each $k \geq 0$, since x lies strictly between c_k and c_{k+1} we have $x \neq c_k$, that is $\frac{r}{s} \neq \frac{p_k}{q_k}$, and so $rq_k \neq sp_k$. It follows that for every $k \geq 0$ we have

$$0 < \tfrac{1}{sq_k} \leq \tfrac{|rq_k - sp_k|}{sq_k} = \left|\tfrac{r}{s} - \tfrac{p_k}{q_k}\right| = |x - c_k| < |c_{k+1} - c_k| = \tfrac{1}{q_{k+1}q_k} < \tfrac{1}{{q_k}^2}$$

and so $0 < \frac{1}{s} < \frac{1}{q_k}$. But this is not possible since $q_k \to \infty$ as $k \to \infty$, and so $x \in \mathbf{R} \setminus \mathbf{Q}$.

Next, let us show that if $x = [a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$ then the terms a_n are uniquely determined by the formula in the statement of the theorem. Let $a_0 \in \mathbf{Z}$ and let $a_k \in \mathbf{Z}^+$ for $k \geq 1$ and let $\{x_n\}$ be the sequence given by $x_0 = x$ and $x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor}$ for $k \geq 1$. For all $n \geq 1$ we have $[a_0, a_1, \cdots, a_n] = a_0 + \frac{1}{[a_1, a_2, \cdots, a_n]}$. Taking the limit on both sides as $n \to \infty$ we obtain $[a_0, a_1, \cdots] = a_0 + \frac{1}{[a_1, a_2, \cdots]}$. Since $[a_0, a_1, \cdots] > a_0$ and $[a_1, a_2, \cdots] > a_1$ (by Part (5) of the above theorem) we have

$$a_0 < [a_0, a_1, \cdots] = a_0 + \frac{1}{[a_1, a_2, \cdots]} < a_0 + \frac{1}{a_1} \leq a_0 + 1$$

so that $a_0 < x_0 < a_0 + 1$ and hence $a_0 = \lfloor x_0 \rfloor$.. Also, since $[a_0, a_1, \cdots] = a_0 + \frac{1}{[a_1, a_2, \cdots]}$, we have $[a_1, a_2, \cdots] = \frac{1}{[a_0, a_1, \cdots] - a_0} = \frac{1}{x_0 - \lfloor x_0 \rfloor} = x_1$. Repeating the above argument inductively, we find that for all $n \geq 1$ we have $a_n = \lfloor x_n \rfloor$ and $x_n = [a_n, a_{n+1}, a_{n+2}, \cdots]$.

Finally, we show that if $x \in \mathbf{R} \setminus \mathbf{Q}$ and if a_n is given by the formula in the statement of the theorem then we do indeed have $x = [a_0, a_1, \cdots]$. Let $x \in \mathbf{R} \setminus \mathbf{Q}$. Let $x_0 = x$ and for $k \geq 0$ let $a_k = \lfloor x_k \rfloor$ and $x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor}$. Note that $x_0 = x \notin \mathbf{Q}$ and that whenever $x_k \notin \mathbf{Q}$ we have $x_k - \lfloor x_k \rfloor \notin \mathbf{Q}$ and $0 < x_k - \lfloor x_k \rfloor < 1$ and hence, since $x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor}$, we have $x_{k+1} \notin \mathbf{Q}$ and $x_{k+1} > 1$. It follows, by induction, that for all $k \geq 0$ we have $x_k \notin \mathbf{Q}$ and for all $k \geq 1$ we have $x_k > 1$ and $a_k = \lfloor x_k \rfloor \geq 1$. Since $x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor} = \frac{1}{x_k - a_k}$ we have $x_k = a_k + \frac{1}{x_{k+1}}$. Let $a_k' = a_k$ for $0 \leq k \leq n$ and $a_{n+1}' = x_{k+1}$, and let $c_k' = [a_0', a_1', \cdots, a_k'] = \frac{p_k'}{q_k'}$ for $0 \leq k \leq n+1$. Note that $p_k' = p_k$ and $q_k' = q_k$ for $0 \leq k \leq n$, and so $p_{n+1}' = a_{n+1}' p_n' + p_{n-1}' = x_{n+1} p_n + p_{n-1}$ and similarly $q_{n+1}' = x_{n+1} q_n + q_{n-1}$. For $0 \leq k \leq n$ we have

$$[a_0, a_1, \cdots, a_k, x_{k+1}] = [a_0, a_1, \cdots, a_{k-1}, a_k + \tfrac{1}{x_{k+1}}] = [a_0, a_1, \cdots, a_{k-1}, x_k]$$

and hence

$$x = [x_0] = [a_0, x_1] = [a_0, a_1, x_2] = \cdots = [a_0, a_1, \cdots, a_n, x_{n+1}] = \frac{p_{n+1}'}{q_{n+1}'} \text{ and}$$

$$x - c_n = \frac{p_{n+1}'}{q_{n+1}'} - \frac{p_n}{q_n} = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - q_{n-1}p_n}{q_n(x_{n+1}q_n + q_{n-1})} = \frac{(-1)^{n-1}}{q_n(x_{n+1}q_n + q_{n-1})}.$$

Thus $|x - c_n| = \frac{1}{q_n(x_{n+1}q_n + q_{n-1})} < \frac{1}{2{q_k}^2} \longrightarrow 0$ so that $x = [a_0, a_1, a_2, \cdots]$, as required.

1.8 Example: Express $\sqrt{14}$ as a continued fraction.

Solution: We let $x_0 = x = \sqrt{14}$ then calculate some terms in the sequences $\{x_n\}$ and $\{a_n\}$ using the recursion formulas $a_k = \lfloor a_k \rfloor$ and $x_{k+1} = \frac{1}{x_k - a_k}$.

k	x_k	a_k
0	$\sqrt{14}$	3
1	$\frac{1}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{5}$	1
2	$\frac{5}{\sqrt{14}-2} = \frac{\sqrt{14}+2}{2}$	2
3	$\frac{2}{\sqrt{14}-2} = \frac{\sqrt{14}+2}{5}$	1
4	$\frac{5}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{1}$	6
5	$\frac{1}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{5}$	1

We see that the values of x_k begin to repeat with period 4 so that $x_{k+4} = x_k$ and $a_{k+4} = a_k$ for all $k \geq 1$. Thus we have

$$\sqrt{14} = [3, 1, 2, 1, 6, 1, 2, 1, 6, \cdots] = \left[3, \overline{1, 2, 1, 6}\right].$$

1.9 Example: Let $x \in \mathbf{R} \setminus \mathbf{Q}$ with $x > 1$. Say $x = [a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$. Since $x > 1$ we have $a_0 = \lfloor x \rfloor \geq 1$. For all $n \geq 0$, note that $[0, a_0, a_1, \cdots, a_n] = \frac{1}{[a_0, a_1, \cdots, a_n]}$. By taking the limit on both sides we obtain $[0, a_0, a_1, a_2, \cdots] = \frac{1}{[a_0, a_1, a_2, \cdots]}$. It follows that $\frac{1}{x} = [0, a_0, a_1, a_2, \cdots]$. Also note that the convergents of x, given by $c_n = [a_0, a_1, \cdots, a_n]$, and the convergents of $\frac{1}{x}$, given by $d_n = [0, a_0, a_1, \cdots, a_{n-1}]$, are related by $d_0 = 0$ and $d_{n+1} = \frac{1}{c_n}$ for all $n \geq 0$.

1.10 Theorem: *Let $x = [a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$. For $n \geq 0$, let $c_n = [a_0, a_1, \cdots, a_n] = \frac{p_n}{q_n}$. Let $r \in \mathbf{Z}$ and $s \in \mathbf{Z}^+$. Then*
(1) for all $k \geq 0$, if $|sx - r| < |q_k x - p_k|$ then $s \geq q_{k+1}$,
(2) for all $k \geq 0$, if $\left|x - \frac{r}{s}\right| < \left|x - \frac{p_k}{q_k}\right|$ then $s > q_k$, and
(3) if $\left|x - \frac{r}{s}\right| < \frac{1}{2s^2}$ then $\frac{r}{s} = c_k$ for some $k \geq 0$.

Proof: To prove Part (1) let $k \geq 0$, suppose that $|sx_k - r| < |q_k x - p_k|$ and suppose, for a contradiction, that $s < q_{k+1}$. Note that to get $(r, s) = u(p_k, q_k) + v(p_{k+1}, q_{k+1})$ we need

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{pmatrix}^{-1} \begin{pmatrix} r \\ s \end{pmatrix} = \frac{1}{p_k q_{k+1} - q_k p_{k+1}} \begin{pmatrix} q_{k+1} & -p_{k+1} \\ -q_k & p_k \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \\ &= \frac{-1}{(-1)^k} \begin{pmatrix} q_{k+1} & -p_{k+1} \\ -q_k & p_k \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = (-1)^k \begin{pmatrix} -q_{k+1} r + p_{k+1} s \\ -q_{k+1} r + p_{k+1} s \end{pmatrix}.\end{aligned}$$

Thus we choose $u = (-1)^k(-q_{k+1}r + p_{k+1}s)$ and $v = (-1)^k(-q_{k+1}r + p_{k+1}s)$. Note that $u \in \mathbf{Z}$ and $v \in \mathbf{Z}$ and we have $r = up_k + vp_{k+1}$ and $s = uq_k + vq_{k+1}$. We claim that $u \neq 0$. Suppose, for a contradiction, that $u = 0$. Then we have $s = vq_{k+1}$ which implies that $v > 0$ (since $s > 0$ and $q_{k+1} > 0$) and hence that $s \geq q_{k+1}$. This contradicts our assumption that $s < q_{k+1}$, and so we have $u \neq 0$, as claimed. We claim that $v \neq 0$. Suppose, for a contradiction, that $v = 0$. Then we have $r = up_k$ and $s = uq_k$, and so $|sx - r| = |up_k x - uq_k| = |u||p_k x - q_k| \geq |p_k x - q_k|$. This contradicts our assumption that $|sx - r| < |p_k x - q_k|$, and so we have $v \neq 0$, as claimed. Note that u and v have opposite signs (that is $uv < 0$) because if we had $u > 0$ and $v > 0$ then we would have $s = uq_k + vq_{k+1} > q_{k+1}$, and if we have $u < 0$ and $v < 0$ then we would have $s = uq_k + vq_{k+1} < 0$. Note that $(q_k x - p_k)$ and $(q_{k+1}x - p_{k+1})$ have opposite signs because x lies between $c_k = \frac{p_k}{q_k}$ so that $x - c_k$ and $x - c_{k+1}$ have opposite signs. Thus, since $(q_k x - p_k)u$ and $(q_{k+1}x - p_{k+1})v$ have the same sign, we have

$$\begin{aligned}|sx - c| &= \left|(uq_k + vq_{k+1})x - (up_k + vp_{k+1})\right| = \left|(q_k x - p_k)u + (q_{k+1}x - p_{k+1})v\right| \\ &= |q_k x - p_k||u| + |q_{k+1}x - p_{k+1}||v| > |q_k x - p_k|.\end{aligned}$$

This contradicts the fact that $|sx - r| < |q_k x - p_k|$ and completes the proof of Part (1).

To prove Part (2) let $k \geq 0$, suppose that $\left|x - \frac{r}{s}\right| < \left|x - \frac{p_k}{q_k}\right|$ and suppose, for a contradiction, that $s \leq q_k$. Then we have

$$|sx - r| = s\left|x - \tfrac{r}{s}\right| < s\left|x - \tfrac{p_k}{q_k}\right| \leq q_k\left|x - \tfrac{p_k}{q_k}\right| = |q_k x - p_k|.$$

But then, by Part (2), we have $s \geq q_{k+1}$ so that $s > q_k$, giving the desired contradiction.

To prove Part (3), suppose that $\left|x - \frac{r}{s}\right| < \frac{1}{2s^2}$. Since $q_0 = 1$ and $\{q_n\}$ is increasing with $q_n \longrightarrow \infty$ as $n \to \infty$, we can choose $k \geq 0$ so that $q_k \leq s < q_{k+1}$. We claim that $\frac{r}{s} = c_k$. Suppose, for a contradiction, that $\frac{r}{s} \neq c_k$. Since $s < q_{k+1}$ it follows from Part (1) that $|q_k x - p_k| \leq |sx - r|$, and so $\left|x - \frac{p_k}{q_k}\right| = \frac{1}{q_k}|q_k x - p_k| \leq \frac{1}{q_k}|sx - r| = \frac{s}{q_k}\left|x - \frac{r}{s}\right| < \frac{s}{q_k} \cdot \frac{1}{2s^2} = \frac{1}{2sq_k}$. Since $\frac{r}{s} \neq c_k$, that is $\frac{r}{s} \neq \frac{p_k}{q_k}$, we have $rq_k - sp_k \neq 0$ and so $\left|\frac{r}{s} - \frac{p_k}{q_k}\right| = \frac{|rq_k - sp_k|}{sq_k} \geq \frac{1}{sq_k}$. Thus we have

$$\frac{1}{sq_k} \leq \left|\tfrac{r}{s} - \tfrac{p_k}{q_k}\right| \leq \left|\tfrac{r}{s} - x\right| + \left|x - \tfrac{p_k}{q_k}\right| < \frac{1}{2s^2} + \frac{1}{2sq_k}.$$

Subtracting $\frac{1}{2sq_k}$ from both sides gives $\frac{1}{2sq_k} < \frac{1}{2s^2}$ so that $s < q_k$. This contradicts the fact that $q_k \leq s$, and so we have $\frac{r}{s} = c_k$, as claimed.

1.11 Corollary: *Let $d \in \mathbf{Z}^+$ be a non-square and let $r, s \in \mathbf{Z}^+$. If $|r^2 - ds^2| \leq \sqrt{d}$ then $\frac{r}{s}$ is equal to one of the convergents of $\sqrt{d}$.*

Proof: Suppose that $|r^2 - ds^2| \leq \sqrt{d}$. We consider two cases. Case 1: suppose that $0 < r^2 - ds^2 \leq \sqrt{d}$. Since $(r + s\sqrt{d})(r - s\sqrt{d}) = r^2 - ds^2 > 0$, we have $r - s\sqrt{d} > 0$, that is $r > s\sqrt{d}$. It follows that $0 < \frac{r}{s} - \sqrt{d} = \frac{r - s\sqrt{d}}{s} = \frac{r^2 - ds^2}{s(r + s\sqrt{d})} \leq \frac{\sqrt{d}}{s(r + s\sqrt{d})} < \frac{\sqrt{d}}{s(s\sqrt{d} + s\sqrt{d})} = \frac{1}{2s^2}$. By Part (3) of the previous theorem, $\frac{r}{s}$ must be equal to one of the convergents of $\sqrt{d}$.

Case 2: suppose that $-\sqrt{d} < r^2 - ds^2 < 0$. Since $(r + s\sqrt{d})(r - s\sqrt{d}) = r^2 - ds^2 < 0$ we have $r - s\sqrt{d} < 0$ so that $r < s\sqrt{d}$. It follows that

$$0 < \frac{s}{r} - \frac{1}{\sqrt{d}} = \frac{s\sqrt{d} - r}{r\sqrt{d}} = \frac{s^2 d - r^2}{r\sqrt{d}(s\sqrt{d} + r)} < \frac{\sqrt{d}}{r\sqrt{d}(s\sqrt{d} + r)} < \frac{\sqrt{d}}{r\sqrt{d}(r + r)} = \frac{1}{2r^2}.$$

By Part (3) of the previous theorem, $\frac{s}{r}$ must be equal to one of the convergents of $\frac{1}{\sqrt{d}}$. By Example 1.9, it follows that $\frac{r}{s}$ is equal to one of the convergents of $\sqrt{d}$.

1.12 Definition: A **quadratic irrational** is an irrational number which is a root of a quadratic polynomial with coefficients in $\mathbf{Z}$.

1.13 Theorem: *The quadratic irrational numbers are the numbers of the form $x = \frac{r + \sqrt{d}}{s}$ for some non-square $d \in \mathbf{Z}^+$ and some $r, s \in \mathbf{Z}$ with $s \neq 0$ and $s \big| (r^2 - d)$.*

Proof: Suppose that $x = \frac{r + \sqrt{d}}{s}$ where $d \in \mathbf{Z}^+$ is a non-square and $r, s \in \mathbf{Z}$ with $s \neq 0$ and $s \big| (r^2 - d)$. Then x is irrational and we have $sx - r = \sqrt{d}$ so that $s^2 x^2 - 2rs\, x + r^2 = d$, and so x is a root of $f(x) = sx^2 - 2rx + \frac{r^2 - d}{s} \in \mathbf{Z}[x]$.

Conversely, let x be an irrational number which is a root of $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbf{Z}$ with $a \neq 0$. By the Quadratic Formula, we have $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Let $d = b^2 - 4ac \in \mathbf{Z}$. Since x is irrational number, $d \geq 0$ and d is not a square. When $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ we have $x = \frac{r + \sqrt{d}}{s}$ for $r = -b$ and $s = 2a$. When $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ we have $x = \frac{r + \sqrt{d}}{s}$ for $r = b$ and $s = -2a$. In either case, $s \neq 0$, $r^2 - d = 4ac$ and $s \big| (r^2 - d)$.

1.14 Definition: For $x = \frac{r + \sqrt{d}}{s}$ where $d \in \mathbf{Z}^+$ is non-square and $r, s \in \mathbf{Z}$ with $s \neq 0$ and $s \big| (r^2 - d)$, we define the **conjugate** of x to be $\overline{x} = \frac{r - \sqrt{d}}{s}$.

1.15 Theorem: *Let $x = [a_0, a_1, a_2, \cdots]$ where $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$. Then*
(1) $\{a_n\}$ is eventually periodic $\iff$ x is a quadratic irrational, and
(2) $\{a_n\}$ is purely periodic $\iff$ x is a quadratic irrational with $x > 1$ and $-1 < \overline{x} < 0$.

Proof: I may include a proof later.

1.16 Theorem: *Let $d \in \mathbf{Z}^+$ be a non-square. Let $\sqrt{d} = [a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_k \in \mathbf{Z}^+$ for $k \geq 1$. Let ℓ be the minimum period of $\{a_n\}$. Let $c_n = [a_0, a_1, \cdots, a_n] = \frac{p_n}{q_n}$. Let $x_0 = \sqrt{d}$ and $x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor}$ for $k \geq 0$. Write $x_k = \frac{r_k + \sqrt{d}}{s_k}$. Then*
(1) we have $a_\ell = 2a_0$ so that $\sqrt{d} = \big[a_0, \overline{a_1, a_2, \cdots, a_{\ell-1}, 2a_0}\,\big]$,
(2) the sequence $\{s_n\}$ is purely periodic with $s_k = 1 \iff \ell \big| k$,
(3) for all $k \geq 0$ we have ${p_k}^2 - d\,{q_k}^2 = (-1)^k s_{k+1}$, and
(4) the smallest unit u in $\mathbf{Z}[\sqrt{d}]$ with $u > 1$ is equal to $u = p_{\ell-1} + q_{\ell-1}\sqrt{d}$ and we have

$$u^k = p_{k\ell-1} + q_{k\ell-1}\sqrt{d} \text{ for all } k \in \mathbf{Z}^+.$$

Proof: I may include a proof later.

1.17 Example: Find the smallest unit $u \in \mathbf{Z}[\sqrt{19}\,]$ with $u > 1$.

Solution: We find some terms in the sequences $\{x_n\}$ and $\{a_n\}$ using the recursion formulas $x_0 = x = \sqrt{19}$ and $a_k = \lfloor x_k \rfloor$ and $x_{k+1} = \frac{1}{x_k - a_k}$ for $k \geq 0$, and we find some terms in the sequences $\{p_n\}$ and $\{q_n\}$ using the recursion formulas $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $q_0 = 1$ and $q_1 = a_1$ and $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$ for $k \geq 2$, and we calculate the norms $N_k = N(p_k + q_k\sqrt{d}) = {p_k}^2 - d{q_k}^2$.

k	x_k	a_k	p_k	q_k	N_k
0	$\sqrt{19}$	4	4	1	-3
1	$\frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3}$	2	9	2	5
2	$\frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5}$	1	13	3	-2
3	$\frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2}$	3	48	11	5
4	$\frac{2}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{5}$	1	61	14	-3
5	$\frac{5}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{3}$	2	170	39	1
6	$\frac{3}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{1}$	8			

From the table, we see that $\sqrt{19} = \left[4, \overline{2, 1, 3, 1, 2, 8}\right]$, the period is $\ell = 6$, and that the smallest unit $u \in \mathbf{Z}[\sqrt{19}\,]$ with $u > 1$ is equal to $u = 170 + 39\sqrt{19}$.