PMATH 450/650 Solutions to the Exercises for Chapter 2

1: Let $f_n : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be measurable functions for $n \in \mathbf{Z}^+$, and let $a \in \mathbf{R}$.

(a) Show that $\left\{x \in A \,\middle|\, \liminf\limits_{n\to\infty} f_n(x) > a\right\} = \bigcup\limits_{m=1}^{\infty} \bigcup\limits_{\ell=1}^{\infty} \bigcap\limits_{n=\ell}^{\infty} \left\{x \in A \middle| f_n(x) \geq a + \frac{1}{m}\right\}$.

Solution: Let $S = \left\{x \in A \,\middle|\, \liminf\limits_{n\to\infty} f_n(x) > a\right\}$ and $T = \bigcup\limits_{m=1}^{\infty} \bigcup\limits_{\ell=1}^{\infty} \bigcap\limits_{n=\ell}^{\infty} \left\{x \in A \middle| f_n(x) \geq a + \frac{1}{m}\right\}$. We have

$$\begin{aligned}
x \in S &\iff \liminf_{n\to\infty} f_n(x) > a \iff \sup\left\{ \inf_{n\geq 1} f_n(x)\,,\, \inf_{n\geq 2} f_n(x)\,,\, \inf_{n\geq 3} f_n(x)\,,\, \cdots \right\} > a \\
&\iff \exists\, m \in \mathbf{Z}^+ \ \sup\left\{ \inf_{n\geq 1} f_n(x)\,,\, \inf_{n\geq 2} f_n(x)\,,\, \inf_{n\geq 3} f_n(x)\,,\, \cdots \right\} > a + \tfrac{1}{m} \\
&\iff \exists\, m \in \mathbf{Z}^+ \ \exists\, \ell \in \mathbf{Z}^+ \ \inf_{n \geq \ell} f_n(x) > a + \tfrac{1}{m} \\
&\implies \exists\, m \in \mathbf{Z}^+ \ \exists\, \ell \in \mathbf{Z}^+ \ \forall\, n \geq \ell \ f_n(x) \geq a + \tfrac{1}{m} \iff x \in T
\end{aligned}$$

and

$$\begin{aligned}
x \in T &\iff \exists\, m \in \mathbf{Z}^+ \ \exists\, \ell \in \mathbf{Z}^+ \ \forall\, n \geq \ell \ f_n(x) \geq a + \tfrac{1}{m} \\
&\iff \exists\, m \in \mathbf{Z}^+ \ \exists\, \ell \in \mathbf{Z}^+ \ \inf_{n \geq \ell} f_n(x) \geq a + \tfrac{1}{m} \\
&\implies \exists\, m \in \mathbf{Z}^+ \ \sup\left\{ \inf_{n\geq 1} f_n(x)\,,\, \inf_{n\geq 2} f_n(x)\,,\, \inf_{n\geq 3} f_n(x)\,,\, \cdots \right\} \geq a + \tfrac{1}{m} \\
&\iff \sup\left\{ \inf_{n\geq 1} f_n(x)\,,\, \inf_{n\geq 2} f_n(x)\,,\, \inf_{n\geq 3} f_n(x)\,,\, \cdots \right\} > a \iff x \in S.
\end{aligned}$$

(b) Show that the set $\left\{x \in A \,\middle|\, \{f_n(x)\} \text{ converges}\right\}$ is measurable.

Solution: Let $B = \left\{x \in A \,\middle|\, \{f_n(x)\} \text{ converges in } \mathbf{R}\right\}$. Then

$$\begin{aligned}
x \in B &\iff \{f_n(x)\} \text{ converges} \iff \{f_n(x)\} \text{ is Cauchy} \\
&\iff \forall\, m \in \mathbf{Z}^+ \ \exists\, n \in \mathbf{Z}^+ \ \forall\, k, l \geq n \ \left|f_k(x) - f_\ell(x)\right| < \tfrac{1}{m} \\
&\iff x \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \bigcap_{\ell=n}^{\infty} \left\{x \in A \,\middle|\, \left|f_k(x) - f_\ell(x)\right| < \tfrac{1}{m}\right\} \\
&\iff x \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \bigcap_{\ell=n}^{\infty} \left\{x \in A \,\middle|\, -\tfrac{1}{m} < f_k(x) - f_\ell(x) < \tfrac{1}{m}\right\} \\
&\iff x \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \bigcap_{\ell=n}^{\infty} {h_{k,\ell}}^{-1}\left(-\tfrac{1}{m}, \tfrac{1}{m}\right)
\end{aligned}$$

where $h_{k,\ell}$ is the measurable function $h_{k,\ell} = f_k - f_\ell$. Thus

$$B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \bigcap_{\ell=n}^{\infty} {h_{k,\ell}}^{-1}\left(-\tfrac{1}{m}, \tfrac{1}{m}\right)$$

which is a measurable set. We remark that if one interprets the statement "$\{f_n\}$ converges" to mean that $\{f_n(x)\}$ converges in $[-\infty, \infty]$, then one must also verify that the sets $C = \left\{x \in A \middle| \lim\limits_{n\to\infty} f_n(x) = \infty\right\}$ and $D = \left\{x \in A \middle| \lim\limits_{n\to\infty} f_n(x) = -\infty\right\}$ are measurable. Since $x \in C \iff \forall\, m \in \mathbf{Z}^+ \ \exists\, n \in \mathbf{Z}^+ \ \forall\, k \geq n \ f_n(x) > m$, we see that $C = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} {f_k}^{-1}(m, \infty]$, which is measurable. Similarly D is measurable.

2: In this problem, you are asked to prove parts of several of the theorems from Chapter 2 in the Lecture Notes. Your proofs should not make use of any theorems from Chapter 2 (you may use theorems from Chapter 1). Let $A = B \cup C$ where $B, C \subseteq \mathbf{R}$ are disjoint and measurable, let $f : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be a function, and let g and h be the restrictions of f to B and to C respectively.

(a) Show that f is measurable if and only if g and h are both measurable.

Solution: Suppose that f is measurable. Let U be any open set in $[-\infty, \infty]$. Since the function f is measurable, the set $f^{-1}(U)$ is measurable, and so the sets $g^{-1}(U) = f^{-1}(U) \cap B$ and $h^{-1}(U) = f^{-1}(U) \cap C$ are both measurable. This shows that the functions g and h are measurable.

Suppose, conversely, that g and h are both measurable. Let U be any open set in $[-\infty, \infty]$. Then the sets $g^{-1}(U)$ and $h^{-1}(U)$ are both measurable, and so the set $f^{-1}(U) = g^{-1}(U) \cup h^{-1}(U)$ is also measurable. This shows that f is measurable.

(b) Suppose that f is a nonnegative simple function. Show that $\displaystyle\int_A f = \int_B g + \int_C h$.

Solution: Write $f = \sum\limits_{k=1}^{n} a_k \chi_{A_k}$ where each $a_k \in [0, \infty)$ and the sets A_k are measurable and disjoint with $A = \bigcup\limits_{k=1}^{n} A_k$. and where $\chi_{A_k} : A \to [0, \infty)$ is the characteristic function for A_k on A. Let $B_k = A_k \cap B$ and $C_k = A_k \cap C$ and note that the sets B_k and C_k are disjoint and measurable with $A_k = B_k \cup C_k$. The restrictions g and h of f are given by $g = \sum\limits_{k=1}^{n} a_k \chi_{B_k}$ and $h = \sum\limits_{k=1}^{n} a_k \chi_{C_k}$ where $\chi_{B_k} : B \to [0, \infty)$ is the characteristic function for B_k on B and $\chi_{C_k} : C \to [0, \infty)$ is the characteristic function for C_k on C. Then

$$\int_B g + \int_C h = \sum_{k=1}^{n} a_k \lambda(B_k) + \sum_{k=1}^{n} a_k \lambda(C_k) = \sum_{k=1}^{n} a_k \big(\lambda(B_k) + \lambda(C_k)\big) = \sum_{k=1}^{n} a_k \lambda(A_k) = \textstyle\int_A f.$$

(c) Suppose that f is a nonnegative measurable function. Show that $\displaystyle\int_A f = \int_B g + \int_C h$.

Solution: Let $\epsilon > 0$. Choose a simple function r on A with $0 \le r \le f$ and $\int_A f \le \int_A r + \epsilon$. Let s and t be the restrictions of r to B and to C. Then s and t are simple functions with $0 \le s \le g$ and $0 \le t \le h$ so we have

$$\int_B g + \int_C h \ge \int_B s + \int_C t = \int_A r \ge \int_A f - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $\int_B g + \int_C h \ge \int_A f$.

Again let $\epsilon > 0$. Choose a simple function s on B with $0 \le s \le g$ such that $\int_B g \le \int_B s + \epsilon$ and choose a simple function t on C with $0 \le t \le h$ such that $\int_C h \le \int_C t + \epsilon$. Define $r : A \to [0, \infty)$ by $r(x) = s(x)$ when $x \in B$ and $r(x) = t(x)$ when $x \in C$ so that r is a simple function on A with $0 \le r \le f$, and s and t are the restrictions of r to B and to C., so we have

$$\int_A f \ge \int_A r = \int_B s + \int_C t \ge \left(\int_B g - \epsilon\right) + \left(\int_C h - \epsilon\right) = \int_B g + \int_C h - 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $\int_A f \ge \int_B g + \int_C h$.

(d) Suppose that f is an integrable function. Show that g and h are integrable and $\displaystyle\int_A f = \int_B g + \int_C h$.

Solution: Note that g^+ and h^+ are the restrictions of f^+ to B and to C, and g^- and h^- are the restrictions of f^- to B and to C. Thus, using Part (c), we have

$$\begin{aligned}\int_A f &= \int_A f^+ - \int_A f^- = \left(\int_B g^+ + \int_C h^+\right) - \left(\int_B g^- + \int_C h^-\right) \\ &= \left(\int_B g^+ - \int_B g^-\right) + \left(\int_C h^+ - \int_C h^-\right) = \int_B g + \int_C h\,.\end{aligned}$$

3: Let $f : A \subseteq \mathbf{R} \to [0, \infty]$ be a nonnegative measurable function.

(a) Show that if $0 < a$ then

$$\lambda\Big(f^{-1}\big((a,\infty]\big)\Big) \le \frac{1}{a}\int_A f\,.$$

Solution: Let $a > 0$. Let $B = f^{-1}(a,\infty]$ and $C = f^{-1}[0,a]$. Then B and C are disjoint and measurable and $A = B \cup C$. Since $f(x) > a$ for all $x \in B$ and $f(x) \ge 0$ for all $x \in C$ we have

$$\int_A f = \int_B f + \int_C f \ge \int_B a + \int_C 0 = a\,\lambda(B)$$

and so $\lambda(B) \le \frac{1}{a}\int_A f$.

(b) Show that if $\int_A f = 0$ then $f = 0$ a.e. in A.

Solution: Suppose that $\int_A f = 0$. Let $B = \big\{x \in A \big| f(x) \ne 0\big\} = f^{-1}(0,\infty]$. For $n \in \mathbf{Z}^+$ let $B_n = f^{-1}\big(\frac{1}{n},\infty\big]$. Then each B_n is measurable with $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} B_n = B$ so we have $\lambda(B) = \lim_{n\to\infty}\lambda(B_n)$. Suppose, for a contradiction, that $\lambda(B) > 0$. Choose $n \in \mathbf{Z}^+$ so that $\lambda(B_n) \ge \frac{1}{2}\lambda(B)$. Then

$$0 = \int_A f \ge \int_{B_n} f \ge \int_{B_n} \tfrac{1}{n} = \tfrac{1}{n}\,\lambda(B_n) \ge \tfrac{1}{2n}\lambda(B) > 0$$

giving the desired contradiction. Thus $\lambda(B) = 0$ and $f(x) = 0$ for all $x \in A \setminus B$, so $f = 0$ a.e. in A.

(c) Show that if $0 < a < \lambda(A) < \infty$ and $f(x) > 0$ for all $x \in A$ then

$$\inf\left\{ \int_B f \,\middle|\, B \subseteq A \text{ is measurable with } \lambda(B) \ge a \right\} > 0\,.$$

Solution: Suppose that $0 < a < \lambda(A) < \infty$ and $f(x) > 0$ for all $x \in A$. For $n \in \mathbf{Z}^+$, let $A_n = f^{-1}\big(\frac{1}{n},\infty\big]$. The sets A_n are measurable with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, and we have $\bigcup_{n=1}^{\infty} A_n = A$, so $\lambda(A) = \lim_{n\to\infty}\lambda(A_n)$. Choose $m \in \mathbf{Z}^+$ so that $\lambda(A_m) \ge \lambda(A) - \frac{a}{2}$.

Let $B \subseteq A$ be any measurable subset of A with $\lambda(B) \ge a$. Since $A_m \subseteq A$ with $\lambda(A_m) \ge \lambda(A) - \frac{a}{2}$ we have $\lambda(A \setminus A_m) = \lambda(A) - \lambda(A_m) \le \frac{a}{2}$, and so

$$a \le \lambda(B) = \lambda(B \cap A_m) + \lambda(B \setminus A_m) \le \lambda(B \cap A_m) + \lambda(A \setminus A_m) \le \lambda(B \cap A_m) + \tfrac{a}{2}$$

and hence $\lambda(B \cap A_m) \ge a - \frac{a}{2} = \frac{a}{2}$. Since $f(x) \ge \frac{1}{m}$ for all $x \in A_m$ and $f(x) \ge 0$ for all $x \in B \setminus A_m$ we have

$$\int_B f = \int_{B\cap A_m} f + \int_{B\setminus A_m} f \ \ge \int_{B\cap A_m} \tfrac{1}{m} + \int_{B\setminus A_m} 0 \ = \tfrac{1}{m}\cdot\lambda(B \cap A_m) \ge \tfrac{1}{m}\cdot\tfrac{a}{2}.$$

Since $\int_B f \ge \frac{a}{2m}$ for every measurable set $B \subseteq A$ with $\lambda(B) = a$, it follows that

$$\inf\left\{ \int_B f \,\middle|\, B \subseteq A \text{ is measurable with } \lambda(B) \ge a \right\} \ge \tfrac{a}{2m} > 0.$$

4: (a) Let $f_n : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be measurable and let $g_n : A \subseteq \mathbf{R} \to [0, \infty]$ be nonnegative and measurable with $\left|f_n\right| \leq g_n$ for all $n \in \mathbf{Z}^+$. Suppose that $\lim\limits_{n\to\infty} f_n$ exists, $\lim\limits_{n\to\infty} g_n$ exists, and $\lim\limits_{n\to\infty} \int_A g_n = \int_A \lim\limits_{n\to\infty} g_n < \infty$. Show that $\int_A \lim\limits_{n\to\infty} f_n = \lim\limits_{n\to\infty} \int_A f_n$.

Solution: Applying Fatou's Lemma to $(g_n + f_n)$ gives

$$\begin{aligned}\int_A \lim_{n\to\infty} g_n + \int_A \lim_{n\to\infty} f_n &= \int_A \liminf_{n\to\infty}(g_n + f_n) \leq \liminf_{n\to\infty} \int_A (g_n + f_n) = \liminf_{n\to\infty} \left(\int_A g_n + \int_A f_n \right) \\ &= \lim_{n\to\infty} \int_A g_n + \liminf_{n\to\infty} \int_A f_n = \int_A \lim_{n\to\infty} g_n + \liminf_{n\to\infty} \int_A f_n\end{aligned}$$

so that

$$\liminf_{n\to\infty} \int_A f_n \geq \int_A \lim_{n\to\infty} f_n \,.$$

Applying Fatou's Lemma to $(g_n - f_n)$ gives

$$\begin{aligned}\int_A \lim_{n\to\infty} g_n - \int_A \lim_{n\to\infty} f_n &= \int_A \liminf_{n\to\infty}(g_n - f_n) \leq \liminf_{n\to\infty} \int_A (g_n - f_n) = \liminf_{n\to\infty} \left(\int_A g_n - \int_A f_n \right) \\ &= \lim_{n\to\infty} \int_A g_n + \liminf_{n\to\infty} \left(- \int_A f_n \right) = \int_A \lim_{n\to\infty} g_n - \limsup_{n\to\infty} \int_A f_n\end{aligned}$$

so that

$$\limsup_{n\to\infty} \int_A f_n \leq \int_A \lim_{n\to\infty} f_n \,.$$

(b) Let $f_n : A \subseteq \mathbf{R} \to [-\infty, \infty]$ be integrable, suppose $\lim\limits_{n\to\infty} f_n$ exists and is integrable, and let $f = \lim\limits_{n\to\infty} f_n$. Show that $\lim\limits_{n\to\infty} \int_A \left|f_n - f\right| = 0$ if and only if $\lim\limits_{n\to\infty} \int_A |f_n| = \int_A |f|$.

Solution: Suppose that $\lim\limits_{n\to\infty} \int_A \left|f_n - f\right| = 0$. Then

$$\left| \int_A |f_n| - \int_A |f| \right| = \left| \int_A |f_n| - |f| \right| \leq \int_A \big| |f_n| - |f| \big| \leq \int_A |f_n - f| \longrightarrow 0 \text{ as } n \to \infty.$$

Suppose, conversely, that $\lim\limits_{n\to\infty} \int_A |f_n| = \int_A |f|$. We apply Part (a) with f_n replaced by $|f - f_n|$ and using $g_n = |f| + |f_n|$. Note that the hypotheses are satisfied because

$$|f - f_n| \leq |f| + |f_n| = g_n \ , \ \lim_{n\to\infty} |f - f_n| = 0 \ , \ \lim_{n\to\infty} g_n = 2|f| \text{ and}$$

$$\begin{aligned}\lim_{n\to\infty} \int_A g_n &= \lim_{n\to\infty} \int_A |f| + |f_n| = \lim_{n\to\infty} \left(\int_A |f| + \int_A |f_n| \right) \\ &= \int_A |f| + \lim_{n\to\infty} \int_A |f_n| = \int_A |f| + \int_A |f| = \int_A 2|f| = \int_A \lim_{n\to\infty} g_n \,.\end{aligned}$$

By Part (a), with f_n replaced by $|f - f_n|$, we have

$$\lim_{n\to\infty} \int_A |f - f_n| = \int_A \lim_{n\to\infty} |f - f_n| = \int_A 0 = 0 \,.$$

5: Let $A \subseteq \mathbf{R}$ be measurable with $\lambda(A) < \infty$ and let Let $f : A \to \mathbf{R}$ be bounded. Define the upper and lower Lebesgue integrals of f on A to be

$$U(f) = \inf\left\{ \int_A s \,\middle|\, s \text{ is a simple function on } A \text{ with } s \geq f \right\} \text{ and}$$
$$L(f) = \sup\left\{ \int_A s \,\middle|\, s \text{ is a simple function on } A \text{ with } s \leq f \right\}.$$

(a) Show that f is measurable if and only if $U(f) = L(f)$ and, in this case, $\displaystyle\int_A f = U(f) = L(f)$.

Solution: Let $A \subseteq \mathbf{R}$ be measurable with $\lambda(A) < \infty$ (or let $A = [a, b]$) and let $f : A \to \mathbf{R}$ be bounded. Choose $c, d \in \mathbf{R}$ so that $f(A) \subseteq [c, d)$.

Suppose that $f : A \subseteq \mathbf{R} \to \mathbf{R}$ is measurable. For all simple functions $r, s : A \to \mathbf{R}$ with $r \leq f \leq s$, we have $\int_A r \leq \int_A s$, and it follows that $L(f) \leq \int_A f \leq U(f)$. Let $n \in \mathbf{Z}^+$. Let $y_k = c + \frac{d-c}{n} k$ for $1 \leq k \leq n$ so that $[c, d)$ is the disjoint union $[c, d) = \bigcup\limits_{k=1}^{n} [y_{k-1}, y_k)$. Let $A_k = f^{-1}[y_{k-1}, y_k)$ so that the sets A_k are disjoint and measurable and $A = \bigcup\limits_{k=1}^{n} A_k$. Let $r_n, s_n : A \to [c, d)$ be the simple functions $r_n = \sum\limits_{k=1}^{n} y_{k-1} \chi_{A_k}$ and $s_n = \sum\limits_{k=1}^{n} y_k \chi_{A_k}$ so that $r_n \leq f \leq s_n$. Since $r_n \leq f$ we have $L(f) \geq \int_A r_n$ and since $s_n \leq f$ we have $U(f) \leq \int_A s_n$.

$$\begin{aligned} 0 \leq U(f) - L(f) &\leq \int_A s_n - \int_A r_n = \sum_{k=1}^{n} y_{k-1} \lambda(A_k) - \sum_{k=1}^{n} y_k \lambda(A_k) \\ &= \sum_{k=1}^{n} (y_k - y_{k-1}) \lambda(A_k) = \sum_{k=1}^{n} \tfrac{d-c}{n} \lambda(A_k) = \tfrac{d-c}{n} \lambda(A) \end{aligned}$$

Since $0 \leq U(f) - L(f) \leq \frac{d-c}{n} \lambda(A)$ for all $n \in \mathbf{Z}^+$ it follows that $U(f) = L(f)$ Since $L(f) \leq \int_A f \leq U(f)$ and $U(f) = L(f)$, we have $\int_A f = U(f) = L(f)$.

Suppose, conversely, that $U(f) = L(f)$. By the definition of $L(f)$ we can choose simple functions r_n' on A with $r_n' \leq f$ such that $L(f) - \frac{1}{n} \leq \int_A r_n'$. Let $r_1 = r_1'$ and for $n \geq 2$ let $r_n = \max(r_{n-1}, r_n')$ to obtain an increasing sequence $\{r_n\}$ of simple functions on A with each $r_n \leq f$ with $L(f) - \frac{1}{n} \leq \int_A r_n$. Let $g(x) = \lim\limits_{n\to\infty} r_n(x)$ (the limit exists since $\{r_n\}$ is increasing and bounded above). Since each r_n is measurable, so is g. Since each $r_n \leq f$, we have $g \leq f$. By the Dominated Convergence Theorem, we have $\int_A g = \lim\limits_{n\to\infty} \int_A r_n = L(f)$. Similarly, construct a decreasing sequence $\{s_n\}$ of simple functions on A with each $s_n \geq f$ such that $\int_A s_n \leq U(f) + \frac{1}{n}$ and let $h = \lim\limits_{n\to\infty} s_n$. Since each s_n is measurable, so is h. Since each $s_n \geq f$, we have $h \geq f$. By the Dominated Convergence Theorem, we have $\int_A h = \lim\limits_{n\to\infty} \int_A s_n = U(f)$. Since $g \leq f \leq h$ we have $h - g \geq 0$. Since $h - g \geq 0$ and $\int_A (h - g) = \int_A h - \int_A g = U(f) - L(f) = 0$, it follows that $h - g = 0$ a.e. in A, and hence $g = h$ a.e. in A. Since $g \leq f \leq h$ and $g = h$ a.e. in A, it follows that $f = g = h$ a.e. in A. Since $f = g$ a.e. in A and g is measurable, it follows that f is measurable.

(b) When $A = [a, b]$, show that if f is Riemann integrable then f is Lebesgue integrable and the two kinds of integral agree.

Solution: Let $A = [a, b] \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$ be bounded. Let $L(f)$ and $U(f)$ be as above, and let

$$L_R(f) = \sup\left\{ \int_A r \,\middle|\, r \text{ is a step function on } A \text{ with } r \leq f \right\}$$
$$U_R(f) = \inf\left\{ \int_A r \,\middle|\, s \text{ is a step function on } A \text{ with } s \leq f \right\}$$

Since every step- function on A is also a simple function on A it follows that $L_R(f) \leq L(f)$ and $U_R(f) \geq U(f)$. Since for all simple functions r and s with $r \leq f \leq s$ we have $\int_A r \leq \int_A s$ it follows that $L(f) \leq U(f)$, and so we have

$$L_R(f) \leq L(f) \leq U(f) \leq U_R(f).$$

Suppose that f is Riemann integrable. Then $\int_a^b f(x)\,dx = L_R(x) = U_R(x)$. Since $L_R(f) = U_R(f)$ and $L_R(f) \leq L(f) \leq U(f) \leq U_R(f)$, it follows that $L_R(f) = L(f) = U(f) = U_R(f)$. Since $U(f) = L(f)$ we have $\int_A f = L(f) = U(f)$ by Part (a).