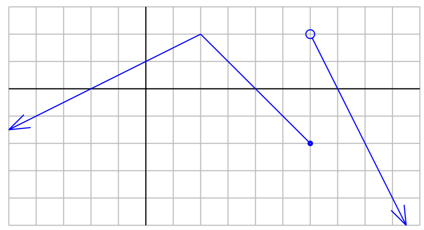
MATH 138 Calculus 2, Solutions to Assignment 1

1: Let $g(x) = \int_0^x f(t)\,dt$ where $f(t)$ is the function whose graph is shown below. Sketch the graph of $y = g(x)$ showing all intercepts, all local maxima and minima, and all points of inflection.



Solution: By the Fundamental Theorem of Calculus, we know that $g'(x) = f(x)$ (except when $x = 6$). When $f(x) > 0$ we have $g'(x) > 0$ so $g(x)$ is increasing and when $f(x) < 0$ we have $g'(x) < 0$ so $g(x)$ is decreasing, and each time $f(x)$ changes sign, $g(x)$ has a local maximum or minimum according to the First Derivative Test. Thus $g(x)$ is decreasing in $(-\infty, -2)$, increasing in $(-2, 4)$, decreasing in $(4, 6)$, increasing in $(6, 7)$, and decreasing in $(7, \infty)$, and $g(x)$ has a local minimum at $x = -2$, a local maximum at $x = 4$, a local minimum at $x = 6$, and a local maximum at $x = 7$. Also, when $f'(x) > 0$ we have $g''(x) > 0$ so $g(x)$ is concave up and when $f'(x) < 0$ we have $g''(x) < 0$ so $g(x)$ is concave down, and each time $f'(x)$ changes sign, $g(x)$ has a point of inflection. Thus $g(x)$ is concave up in $(-\infty, 2)$, concave down in $(2, 6)$, and again concave down in $(6, \infty)$, and $g(x)$ has a point of inflection at $x = 2$.

We can find the exact value of $g(x)$ at each of the points of interest $x = -2, 0, 2, 4, 6, 7$ by interpreting the integral $\int_0^x f(t)\,dt$ as a signed area. For example, $g(2) = \int_0^2 f(t)\,dt$ is the area under $y = f(x)$ between $x = 0$ and $x = 2$, which is equal to 3, so $g(2) = 3$. As another example, $g(6)$ is the area under $y = f(x)$ with $0 \le x \le 4$ minus the are over $y = f(x)$ with $4 \le x \le 6$, so $g(6) = 5 - 2 = 3$. As a final example, $g(-2) = \int_0^{-2} f(t)\,dt = -\int_{-2}^0 f(t)\,dt$, which is the negative of the area under $y = f(x)$ with $-2 \le x \le 0$, so $g(-2) = -1$. To find the x-intercepts of the graph of $g(x)$, we need to determine when $g(x) = 0$ which can also be done by interpreting the integral as a signed area, but it is more convenient to find an explicit formula for $g(x)$ as follows. From the graph of $f(x)$ we can see that

$$f(x) = \begin{cases} 1 + \frac{1}{2}\,x \ , \text{ if } x \le 2 \\ 4 - x \ , \text{ if } 2 \le x \le 6 \\ 14 - 2x \ , \text{ if } 6 < x \,. \end{cases}$$

For $x \le 2$ we have

$$g(x) = \int_0^x f(t)\,dt = \int_0^x 1 + \tfrac{1}{2}\,t\ dx = \Big[t + \tfrac{1}{4}\,t^2\Big]_0^x = x + \tfrac{1}{4}\,x^2\,,$$

for $2 \le x \le 6$ we have

$$\begin{aligned} g(x) &= \int_0^x f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^x f(t)\,dt = g(2) + \int_2^x 4 - t\ dt \\ &= \big(2 + \tfrac{1}{4}\cdot 2^2\big) + \Big[4t - \tfrac{1}{2}\,t^2\Big]_2^x = 3 + \big(4x - \tfrac{1}{2}\,x^2\big) - (8 - 2) = -3 + 4x - \tfrac{1}{2}\,x^2\,, \end{aligned}$$

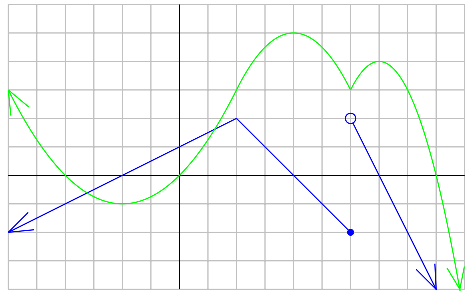
and for $6 \le x$ we have

$$\begin{aligned} g(x) &= \int_0^x f(t)\,dt = \int_0^6 f(t)\,dt + \int_6^x f(t)\,dt = g(6) + \int_6^x 14 - 2x\ dx \\ &= \big(-3 + 4\cdot 6 - \tfrac{1}{2}\cdot 6^2\big) + \Big[14t - t^2\Big]_6^x = 3 + \big(14x - x^2\big) - \big(4\cdot 6 - \cdot 6^2\big) = -45 + 14x - x^2\,. \end{aligned}$$

Thus we have

$$g(x) = \begin{cases} x + \frac{1}{4}\,x^2 \ , \text{ if } x \le 2 \\ -3 + 4x - \frac{1}{2}\,x^2 \ , \text{ if } 2 \le x \le 6 \\ 45 + 14x - x^2 \ , \text{ if } 6 \le x \end{cases}$$

The x-intercepts occur at $x = -4$, $x = 0$ and $x = 9$, and the graph of $y = g(x)$ is shown below in green.



2: (a) Let $f(x) = \dfrac{8x}{2^{3x}}$. Approximate the integral $\displaystyle\int_0^2 f(x)\,dx$ using the Riemann sum for $f(x)$ which uses the right endpoints of 6 equal-sized subintervals.

Solution: The six intervals are of size $\Delta x = \frac{2-0}{6} = \frac{1}{3}$ and the right endpoints are the points $x_k = 0 + k\,\Delta x = \frac{k}{3}$, that is the points $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}$ and 2. We have

$$\begin{aligned}\sum_{k=1}^{n} f(x_k)\Delta_k x &= \left(f\left(\tfrac{1}{3}\right) + f\left(\tfrac{2}{3}\right) + f(1) + f\left(\tfrac{4}{3}\right) + f\left(\tfrac{5}{3}\right) + f(2)\right)\left(\tfrac{1}{3}\right) \\ &= \left(\tfrac{8\cdot 1}{3\cdot 2} + \tfrac{8\cdot 2}{3\cdot 4} + \tfrac{8\cdot 3}{3\cdot 8} + \tfrac{8\cdot 4}{3\cdot 16} + \tfrac{8\cdot 5}{3\cdot 32} + \tfrac{8\cdot 6}{3\cdot 64}\right)\left(\tfrac{1}{3}\right) \\ &= \left(\tfrac{4}{3} + \tfrac{4}{3} + \tfrac{3}{3} + \tfrac{2}{3} + \tfrac{5}{12} + \tfrac{3}{12}\right)\left(\tfrac{1}{3}\right) \\ &= \left(\tfrac{15}{3}\right)\left(\tfrac{1}{3}\right) \\ &= \tfrac{5}{3}\,.\end{aligned}$$

We remark that it can be shown, using methods of Chapter 2, that the exact value of the integral is

$$\int_0^2 f(x)\,dx = \tfrac{21-2\ln 2}{24(\ln 2)^2}\,.$$

(b) Let $f(x) = \dfrac{1}{x}$. Approximate the integral $\displaystyle\int_{1/5}^{13/5} f(x)\,dx$ using the Riemann sum for $f(x)$ which uses the midpoints of 6 equal-sized subintervals.

Solution: Let $f(x) = \frac{1}{x}$. We divide $\left[\frac{1}{5}, \frac{13}{5}\right]$ into 6 equal intervals using $\Delta x = \frac{b-a}{n} = \frac{\frac{13}{5}-\frac{1}{5}}{6} = \frac{2}{5}$. The endpoints of these intervals are given by $x_k = a + \frac{b-a}{n}\,k = \frac{1}{5} + \frac{2}{5}\,k$ so that $x_0, x_1, x_2, \cdots x_6 = \frac{1}{5}, \frac{3}{5}, \frac{5}{5}, \cdots, \frac{13}{5}$. The midpoints are $c_k = \dfrac{x_k + x_{k-1}}{2}$ so that $c_1, c_2, c_3, \cdots, c_6 = \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \cdots, \frac{12}{5}$. We have

$$\begin{aligned}\int_{1/5}^{13/5} f(x)\,dx \cong \sum_{k=1}^{6} f(c_k)\Delta x &= \big(f(c_1) + f(c_2) + \cdots + f(c_6)\big)\left(\tfrac{2}{5}\right) = \tfrac{2}{5}\left(f\left(\tfrac{2}{5}\right) + f\left(\tfrac{4}{5}\right) + \cdots f\left(\tfrac{12}{5}\right)\right) \\ &= \tfrac{2}{5}\left(\tfrac{5}{2} + \tfrac{5}{4} + \tfrac{5}{6} + \cdots + \tfrac{5}{12}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} = \tfrac{60+30+20+15+12+10}{60} = \tfrac{147}{60}\,.\end{aligned}$$

We remark that, by the FTC, the exact value of this integral is

$$\int_{1/5}^{13/5} \frac{dx}{x} = \Big[\ln x\Big]_{1/5}^{13/5} = \ln\tfrac{13}{5} - \ln\tfrac{1}{5} = \ln 13\,,$$

so the above approximation shows that $\ln 13 \cong \frac{147}{60}$.

3: (a) Evaluate $\displaystyle\int_1^3 x^3 - 3x\,dx$ by finding the limit of a sequence of Riemann sums.

Solution: Let $f(x) = x^3 - 3x$, $a = 1$, $b = 3$, $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, and $x_k = a + \frac{b-a}{n}\,k = 1 + \frac{2}{n}\,k$. Then

$$\begin{aligned}
\int_1^3 x^3 - 3x\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n f\big(1+\tfrac{2}{n}\,k\big)\big(\tfrac{2}{n}\big)\\
&= \lim_{n\to\infty}\sum_{k=1}^n \Big(\big(1+\tfrac{2}{n}\,k\big)^3 - 3\big(1+\tfrac{2}{n}\,k\big)\Big)\big(\tfrac{2}{n}\big)\\
&= \lim_{n\to\infty}\sum_{k=1}^n \Big(\big(1+\tfrac{6}{n}\,k+\tfrac{12}{n^2}\,k^2+\tfrac{8}{n^3}\,k^3\big) - 3\big(1+\tfrac{2}{n}\,k\big)\Big)\big(\tfrac{2}{n}\big)\\
&= \lim_{n\to\infty}\sum_{k=1}^n \Big(-2+\tfrac{12}{n^2}\,k^2+\tfrac{8}{n^3}\,k^3\Big)\big(\tfrac{2}{n}\big)\\
&= \lim_{n\to\infty}\sum_{k=1}^n \Big(-\tfrac{4}{n}+\tfrac{24}{n^3}\,k^2+\tfrac{16}{n^4}\,k^3\Big)\\
&= \lim_{n\to\infty}\Big(-\tfrac{4}{n}\sum_{k=1}^n 1+\tfrac{24}{n^3}\sum_{k=1}^n k^2+\tfrac{16}{n^4}\sum_{k=1}^n k^3\Big)\\
&= \lim_{n\to\infty}\Big(-\tfrac{4}{n}\,n+\tfrac{24}{n^3}\,\tfrac{n(n+\frac{1}{2})(n+1)}{3}+\tfrac{16}{n^4}\,\tfrac{n^2(n+1)^2}{4}\Big)\\
&= -4+\tfrac{24}{3}+\tfrac{16}{4} = 8\,.
\end{aligned}$$

(b) Evaluate $\displaystyle\int_0^1 e^x\,dx$ by finding the limit of a sequence of Riemann sums.

Solution: Let $f(x) = e^x$ and use $a = 0$, $b = 1$, $\Delta x = \frac{b-a}{n} = \frac{1}{n}$ and $x_k = a + \frac{b-a}{n}\,k = \frac{k}{n}$. Then

$$\int_0^1 e^x\,dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n e^{k/n}\cdot\tfrac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^n (e^{1/n})^k = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{(e^{1/n})^{n+1}-e^{1/n}}{e^{1/n}-1}$$

where, at the last step, we used the formula $\sum\limits_{k=1}^n r^k = \frac{r^{n+1}-r}{r-1}$ for the sum of a geometric series. We have

$$\lim_{n\to\infty}(e^{1/n})^{n+1} = \lim_{n\to\infty} e^{(n+1)/n} = e^1\ ,\ \text{and}\ \lim_{n\to\infty} e^{1/n} = e^0 = 1$$

and by replacing $\frac{1}{n}$ by x and then using l'Hôpital's Rule, we have

$$\lim_{n\to\infty} n\,(e^{1/n}-1) = \lim_{n\to\infty}\frac{e^{1/n}-1}{1/n} = \lim_{x\to 0^+}\frac{e^x-1}{x} = \lim_{x\to 0^+}\frac{e^x}{1} = e^0 = 1\,.$$

and so

$$\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{(e^{1/n})^{n+1}-e^{1/n}}{n\,(e^{1/n}-1)} = \frac{e-1}{1} = e-1\,.$$

4: (a) Find $g'(1)$ where $g(x) = \displaystyle\int_{3x-3}^{x^2+1} \sqrt{1+t^3}\,dt$.

Solution: Let $u(x) = x^2+1$ and let $v(x) = 3x-3$. Also, let $f(t) = \sqrt{1+t^3}$ and let $F(u) = \displaystyle\int_0^u \sqrt{1+t^3}\,dt$ so that $F'(u) = f(u)$, by the FTC. Then

$$g(x) = \int_{3x-3}^{x^2+1} \sqrt{1+t^3}\,dt = \int_0^{x^2+1} \sqrt{1+t^3}\,dt - \int_0^{3x-3} \sqrt{1+t^3}\,dt = F\big(u(x)\big) - F\big(v(x)\big)$$

and so $g'(x) = F'\big(u(x)\big)u'(x) - F'\big(v(x)\big)v'(x) = f\big(u(x)\big)(2x) - f\big(v(x)\big)(3) = 2x\,f(x^2+1) - 3\,f(3x-3)$. Put in $x=1$ to get $g'(1) = 2f(2) - 3f(0) = 2\sqrt{1+8} - 3\sqrt{1+0} = 6-3 = 3$.

(b) Find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$.

Solution: Let $f(x) = \frac{1}{1+x}$ and let X_n be the partition of $[0,1]$ into n equal-sized subintervals so $x_{n,k} = \frac{k}{n}$ and $\Delta_{n,k}x = \frac{1}{n}$. Then by the FTC we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n,k})\Delta_{n,k}x = \int_0^1 \frac{dx}{1+x} = \Big[\ln(1+x)\Big]_0^1 = \ln 2\,.$$