MATH 138 Calculus 2, Solutions to Assignment 2

1: (a) Find $\displaystyle\int_0^{\ln 3} \frac{e^x\,dx}{1+e^x}$.

Solution: Make the substitution $u = e^x$ so $du = e^x\,dx$. Then

$$\int_0^{\ln 3} \frac{e^x\,dx}{1+e^x} = \int_1^3 \frac{du}{1+u} = \Big[\ln(1+u)\Big]_1^3 = \ln 4 - \ln 2 = \ln 2\,.$$

(b) Find $\displaystyle\int_1^3 \frac{\sqrt{x}}{x+1}\,dx$.

Solution: Make the substitution $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int_1^3 \frac{\sqrt{x}\,dx}{x+1} = \int_1^{\sqrt{3}} \frac{2u^2\,du}{u^2+1} = \int_1^{\sqrt{3}} 2 - \frac{2}{u^2+1}\,du$$
$$= \Big[2u - 2\tan^{-1}u\Big]_1^{\sqrt{3}} = \big(2\sqrt{3} - \tfrac{2\pi}{3}\big) - \big(2 - \tfrac{\pi}{2}\big)2\sqrt{3} - 2 - \tfrac{\pi}{6}\,.$$

(c) Find $\displaystyle\int_0^{\pi/6} \sqrt{2\sin x}\,\cos^3 x\,dx$.

Solution: Make the substitution $u = \sin x$ so $du = \cos x\,dx$. Then

$$\int_0^{\pi/6} \sqrt{2\sin x}\,\cos^3 x\,dx = \int_0^{\pi/6} \sqrt{2\sin x}\,(1-\sin^2 x)\cos x\,dx = \int_0^{1/2} \sqrt{2u}\,(1-u^2)\,du$$
$$= \sqrt{2}\int_0^{1/2} u^{1/2} - u^{5/2}\,du = \sqrt{2}\left[\tfrac{2}{3}\,u^{3/2} - \tfrac{2}{7}\,u^{7/2}\right]_0^{1/2} = \sqrt{2}\left(\tfrac{2}{3\cdot 2\sqrt{2}} - \tfrac{2}{7\cdot 8\sqrt{2}}\right) = \tfrac{1}{3} - \tfrac{1}{28} = \tfrac{25}{84}\,.$$

2: (a) Find $\displaystyle\int_1^e (\ln x)^2\,dx$.

Solution: Integrate by parts using $u = (\ln x)^2$, $du = \dfrac{2\ln x}{x}\,dx$, $v = x$ and $dv = dx$ to get

$$\int_1^e (\ln x)^2\,dx = \left[x(\ln x)^2 - \int 2\ln x\,dx\right]_1^e = \Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_1^e = (e - 2e + 2e) - 2 = e - 2\,.$$

(b) Find $\displaystyle\int_0^3 \frac{x^2\,dx}{(x+1)^{3/2}}$.

Solution: Make the substitution $u = x+1$ so $x = u-1$ and $dx = du$. Then

$$\int_0^3 \frac{x^2\,dx}{(x+1)^{3/2}} = \int_1^4 \frac{(u-1)^2}{u^{3/2}}\,du = \int_1^4 \frac{u^2-2u+1}{u^{3/2}}\,du = \int_1^4 u^{1/2} - 2u^{-1/2} + u^{-3/2}\,du$$
$$= \left[\tfrac{2}{3}\,u^{3/2} - 4u^{1/2} - 2u^{-1/2}\right]_1^4 = \big(\tfrac{16}{3} - 8 - 1\big) - \big(\tfrac{2}{3} - 4 - 2\big) = \tfrac{5}{3}\,.$$

(c) Find $\displaystyle\int_0^{\pi/4} \tan^5 x\,dx$.

Solution: Make the substitution $u = \sec x$ so $du = \sec x\tan x\,dx$. Then

$$\int_0^{\pi/4} \tan^5 x\,dx = \int_0^{\pi/4} \frac{(\sec^2 x - 1)^2}{\sec x}\,\sec x\tan x\,dx = \int_1^{\sqrt{2}} \frac{(u^2-1)^2}{u}\,du = \int_1^{\sqrt{2}} u^3 - 2u + \frac{1}{u}\,du$$
$$= \left[\tfrac{1}{4}\,u^4 - u^2 + \ln u\right]_1^{\sqrt{2}} = \big(1 - 2 + \ln\sqrt{2}\big) - \big(\tfrac{1}{4} - 1\big) = \tfrac{1}{2}\ln 2 - \tfrac{1}{4}\,.$$

3: (a) Find $\displaystyle\int_0^{\sqrt{3}} \frac{x^2\,dx}{(4-x^2)^{3/2}}$.

Solution: Make the substitution $2\sin\theta = x$ so $2\cos\theta = \sqrt{4-x^2}$ and $2\cos\theta\,d\theta = dx$ to get

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{x^2\,dx}{(4-x^2)^{3/2}} &= \int_0^{\pi/3} \frac{(2\sin\theta)^2 2\cos\theta\,d\theta}{(2\cos\theta)^3} = \int_0^{\pi/3} \tan^2\theta\,d\theta \\ &= \int_0^{\pi/3} \sec^2\theta - 1\;d\theta = \Big[\tan\theta - \theta\Big]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}\,.\end{aligned}$$

(b) Find $\displaystyle\int_0^1 x^2\sqrt{2-x^2}\,dx$.

Solution: Make the substitution $\sqrt{2}\sin\theta = x$ so that $\sqrt{2}\cos\theta = \sqrt{2-x^2}$ and $\sqrt{2}\cos\theta\,d\theta = dx$ to get

$$\begin{aligned}\int_0^1 x^2\sqrt{2-x^2}\,dx &= \int_0^{\pi/4} (\sqrt{2}\sin\theta)^2\cdot\sqrt{2}\cos\theta\cdot\sqrt{2}\cos\theta\,d\theta = \int_0^{\pi/4} 4\sin^2\theta\cos^2\theta\;d\theta \\ &= \int_0^{\pi/4} \sin^2 2\theta\;d\theta = \int_0^{\pi/4} \tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta\;d\theta = \Big[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\Big]_0^{\pi/4} = \tfrac{\pi}{8}\,.\end{aligned}$$

(c) Find $\displaystyle\int \frac{dx}{(x^2+1)^2}$.

Solution: Make the substitution $\tan\theta = x$ so that $\sec\theta = \sqrt{x^2+1}$ and $\sec^2\theta\,d\theta = dx$. Then

$$\begin{aligned}\int \frac{dx}{(x^2+1)^2} &= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta = \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\;d\theta \\ &= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + c = \tfrac{1}{2} + \tfrac{1}{2}\sin\theta\cos\theta + c = \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\,\frac{x}{x^2+1} + c\,.\end{aligned}$$

4: (a) Find $\displaystyle\int e^{2x}\cos x\;dx$ and find $\displaystyle\int_0^{\pi/2} e^{2x}\cos x\;dx$.

Solution: Let $I=\displaystyle\int e^{2x}\cos x\;dx$. Integrate by parts twice, first using $u_1=e^{2x}$, $du_1=2e^{2x}\,dx$, $v_1=\sin x$ and $dv_1=\cos x\,dx$, and then using $u_2=2e^{2x}$, $du_2=4e^{2x}\,dx$, $v_2=-\cos x$ and $dv_2=\sin x\,dx$ to get

$$\begin{aligned} I=\int e^{2x}\cos x\;dx &= u_1v_1-\int v_1du_1 = e^{2x}\sin x-\int 2e^{2x}\sin x\,dx = e^{2x}\sin x-\Big(u_2v_2-\int v_2\,du_2\Big)\\ &= e^{2x}\sin x-\Big(-2e^{2x}\cos x+\int 4e^{2x}\cos x\,dx\Big)=e^{2x}\big(\sin x+2\cos x\big)-4I\,. \end{aligned}$$

so that $5I=e^{2x}\big(\sin x+2\cos x\big)+b$, that is $I=\frac{1}{5}\,e^{2x}\big(\sin x+2\cos x\big)+c$. Thus we have

$$\int e^{2x}\cos x\;dx=\tfrac{1}{5}\,e^{2x}\big(\sin x+2\cos x\big)+c$$

and

$$\int_0^{\pi/2} e^{2x}\cos x\;dx=\Big[\tfrac{1}{5}\,e^{2x}\big(\sin x+2\cos x\big)\Big]_0^{\pi/2}=\tfrac{1}{5}\,e^{\pi}-\tfrac{2}{5}\,.$$

(b) Find $\displaystyle\int x^2\sin^{-1}x\;dx$ and find $\displaystyle\int_0^1 x^2\sin^{-1}x\;dx$.

Solution: Integrate by parts using $u=\sin^{-1}x$, $du=\frac{dx}{\sqrt{1-x^2}}$, $v=\frac{1}{3}x^3$ and $dv=x^2\,dx$, and then make the substitution $w=1-x^2$ so $dw=-2x\,dx$ to get

$$\begin{aligned} \int x^2\sin^{-1}x\;dx &= \tfrac{1}{3}x^3\sin^{-1}x-\int\frac{\frac{1}{3}x^3}{\sqrt{1-x^2}}\,dx=\tfrac{1}{3}x^3\sin^{-1}x-\int\frac{-\frac{1}{6}(1-w)dw}{\sqrt{w}}\\ &= \tfrac{1}{3}x^3\sin^{-1}x+\int\tfrac{1}{6}w^{-1/2}-\tfrac{1}{6}w^{1/2}\;dw=\tfrac{1}{3}x^3\sin^{-1}x+\tfrac{1}{3}w^{1/2}-\tfrac{1}{9}w^{3/2}+c\\ &= \tfrac{1}{3}x^3\sin^{-1}x+\tfrac{1}{3}(1-x^2)^{1/2}-\tfrac{1}{9}(1-x^2)^{3/2}+c \end{aligned}$$

and so

$$\int_0^1 x^2\sin^{-1}x\;dx=\Big[\tfrac{1}{3}x^3\sin^{-1}x+\tfrac{1}{3}(1-x^2)^{1/2}-\tfrac{1}{9}(1-x^2)^{3/2}\Big]_0^1=\big(\tfrac{\pi}{6}\big)-\big(\tfrac{1}{3}-\tfrac{1}{9}\big)=\tfrac{\pi}{6}-\tfrac{2}{9}\,.$$