MATH 138 Calculus 2, Solutions to Assignment 3

1: (a) Find $\displaystyle\int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x}\,dx$.

Solution: We first make the substitution $\tan\theta = x$ so $\sec\theta = \sqrt{1+x^2}$ and $\sec^2\theta\,d\theta = dx$ and then later we make the substitution $u = \sec\theta$ so $du = \sec\theta\tan\theta\,d\theta$. Note that $u = \sec\theta = \sqrt{1+x^2}$, so when $x = \sqrt{3}$ we have $u = 2$ and when $x = \sqrt{8}$ we have $u = 3$. We have

$$\begin{aligned}
\int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x}\,dx &= \int_{\tan^{-1}\sqrt{3}}^{\tan^{-1}\sqrt{8}} \frac{\sec\theta\cdot\sec^2\theta\,d\theta}{\tan\theta} = \int_{\tan^{-1}\sqrt{3}}^{\tan^{-1}\sqrt{8}} \frac{\sec^2\theta}{\sec^2\theta - 1}\,\sec\theta\tan\theta\,d\theta \\
&= \int_2^3 \frac{u^2}{u^2-1}\,du = \int_2^3 1 + \frac{1}{u^2-1}\,du = \int_2^3 1 + \frac{\frac12}{u-1} - \frac{\frac12}{u+1}\,du \\
&= \left[u + \tfrac12\ln\frac{u-1}{u+1}\right]_2^3 = \left(3 + \tfrac12\ln\tfrac12\right) - \left(2 + \tfrac12\ln\tfrac13\right) = 1 + \tfrac12\ln\tfrac32\,.
\end{aligned}$$

(b) Find $\displaystyle\int_1^3 \frac{x^4+3x^3+6}{x^3+4x^2+3x}\,dx$.

Solution: We use long division to obtain

$$\begin{array}{r|l}
 & \qquad\qquad\qquad\quad x - 1 \\
x^3+4x^2+3x & x^4+3x^3+0\,x^2+0x+6 \\
 & \underline{x^4+4x^3+3x^2} \\
 & \qquad -x^3-3x^2+0x+6 \\
 & \qquad \underline{-x^3-4x^2-3x} \\
 & \qquad\qquad\quad x^2+3x+6
\end{array}$$

This shows that $\dfrac{x^4+3x^3+6}{x^3+4x^2+3x} = x-1+\dfrac{x+3x+6}{x^3+4x^2+3x}$. Note that $x^3+4x^2+3x = x(x+1)(x+3)$. In order to get $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+3} = \dfrac{x+3x+6}{x(x+1)(x+3)}$ for all x we need $A(x+1)(x+3)+Bx(x+3)+Cx(x+1) = x^2+3x+6$ for all x. Put in $x=0$ to get $3A = 6$ so $A = 2$, put in $x = -1$ to get $-2B = 4$ so $B = -2$, and put in $x = -3$ to get $6C = 6$ so $C = 1$. Thus

$$\begin{aligned}
\int_1^3 \frac{x^4+3x^3+6}{x^3+4x^2+3x}\,dx &= \int_1^3 x - 1 + \frac{2}{x} - \frac{2}{x+1} + \frac{1}{x+3}\,dx \\
&= \left[\tfrac12 x^2 - x + 2\ln x - 2\ln(x+1) + \ln(x+3)\right]_1^3 \\
&= \left(\tfrac92 - 3 + 2\ln 3 - 2\ln 4 + \ln 6\right) - \left(\tfrac12 - 1 - 2\ln 2 + \ln 4\right) \\
&= 2 + 3\ln 3 - 3\ln 2 = 2 + 3\ln\tfrac32\,.
\end{aligned}$$

2: (a) Find $\displaystyle\int_0^{\pi^2} \sin^2\sqrt{x}\ dx$.

Solution: First we make the substitution $y=\sqrt{x}$ so $y^2=x$ and $2y\,dy=dx$, and then we integrate by parts using $u=y$, $du=dy$, $v=\frac{1}{2}\sin 2y$ and $dv=\cos 2y\,dy$ to get

$$\begin{aligned}\int_0^{\pi^2}\sin^2\sqrt{x}\ dx &= \int_0^\pi 2y\sin^2 y\ dy=\int_0^\pi 2y\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2y\right)\,dy=\int_0^\pi y-y\cos 2y\ dy\\ &=\left[\tfrac{1}{2}y^2-\left(uv-\int v\,du\right)\right]_0^\pi=\left[\tfrac{1}{2}y^2-\tfrac{1}{2}y\sin 2y+\int\tfrac{1}{2}\sin 2y\ dy\right]_0^\pi\\ &=\left[\tfrac{1}{2}y^2-\tfrac{1}{2}y\sin 2y-\tfrac{1}{4}\cos 2y\right]_0^\pi=\left(\tfrac{\pi^2}{2}-\tfrac{1}{4}\right)-\left(-\tfrac{1}{4}\right)=\tfrac{\pi^2}{2}\,.\end{aligned}$$

(b) Find $\displaystyle\int_1^2 \frac{x^3+2}{x^5+2x^3+x}\,dx$.

Solution: To begin with, we find $\displaystyle\int\frac{dx}{(x^2+1)^2}$, which we need later. Let $\tan\theta=x$ so that $\sec\theta=\sqrt{x^2+1}$ and $\sec^2\theta\,d\theta=dx$. Then

$$\begin{aligned}\int\frac{dx}{(x^2+1)^2}&=\int\frac{\sec^2\theta\,d\theta}{\sec^4\theta}=\int\cos^2\theta\,d\theta=\int\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\ d\theta\\ &=\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta+c=\tfrac{1}{2}+\tfrac{1}{2}\sin\theta\cos\theta+c=\tfrac{1}{2}\tan^{-1}x+\tfrac{1}{2}\,\frac{x}{x^2+1}+c\,.\end{aligned}$$

Note that $x^5+2x^3+x=x(x^2+1)^2$. In order to get $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}=\dfrac{x^3+2}{x(x^2+1)^2}$ we need $A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+F)(x^2+1)=x^3+2$. Equate coefficients to get the 5 equations $A+B=0$, $C=1$, $2A+B+D=0$, $C+E=0$ and $A=2$. Solve these to get $A=2$, $B=-2$, $C=1$, $D=-2$ and $E=-1$, and so

$$\begin{aligned}\int_1^2\frac{x^3+2}{x^5+2x^3+x}\,dx&=\int_1^2\frac{2}{x}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2x}{(x^2+1)^2}-\frac{1}{(x^2+1)^2}\ dx\\ &=\left[2\ln x-\ln(x^2+1)+\tan^{-1}x+\frac{1}{x^2+1}-\left(\tfrac{1}{2}\tan^{-1}x+\tfrac{1}{2}\frac{x}{x^2+1}\right)\right]_1^2\\ &=\left[2\ln x-\ln(x^2+1)+\tfrac{1}{2}\tan^{-1}x+\frac{2-x}{2(x^2+1)}\right]_1^2\\ &=\left(2\ln 2-\ln 5+\tfrac{1}{2}\tan^{-1}2\right)-\left(-\ln 2+\tfrac{\pi}{8}+\tfrac{1}{4}\right)\\ &=\ln\tfrac{8}{5}+\tfrac{1}{2}\tan^{-1}2-\tfrac{\pi}{8}-\tfrac{1}{4}\,.\end{aligned}$$

3: (a) Approximate $\displaystyle\int_0^{10} \frac{10\,dx}{x+2}$ using T_5 and find a bound on the error.

Solution: Write $f(x) = \dfrac{10}{x+2}$. Then

$$\begin{aligned}\int_0^{10} f(x)\,dx \cong T_5 &= \frac{(b-a)}{2n}\big(f(0)+2f(2)+2f(4)+2f(6)+2f(8)+f(10)\big)\\ &= \tfrac{10}{10}\big(5+5+\tfrac{10}{3}+\tfrac{5}{2}+2+\tfrac{5}{6}\big) = \tfrac{56}{3}\,.\end{aligned}$$

We have $f'(x) = -\dfrac{10}{(x+2)^2}$ and $f''(x) = \dfrac{20}{(x+2)^3}$, so when $0 \le x \le 10$ we have $|f''(x)| \le \frac{20}{8} = \frac{5}{2}$. So if E is the error in our estimate, then $E \le \dfrac{(b-a)^3}{12n^2}\max|f''(x)| = \frac{10^3}{12\cdot 5^2}\,\frac{5}{2} = \frac{25}{3}$.

(b) Approximate $\displaystyle\int_1^3 \frac{dx}{x}$ using S_4 and find a bound on the error.

Solution: Let $f(x) = 1/x$. Then we have

$$\int_1^3 \frac{dx}{x} \cong S_4 = \frac{2}{12}\big(f(1)+4f\big(\tfrac{3}{2}\big)+2f(2)+4f\big(\tfrac{5}{2}\big)+f(3)\big) = \tfrac{1}{6}\big(1+\tfrac{8}{3}+1+\tfrac{8}{5}+\tfrac{1}{3}\big) = \tfrac{1}{6}\big(\tfrac{33}{5}\big) = \tfrac{11}{10}\,.$$

We have $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{6}{x^4}$ and $f''''(x) = \frac{24}{x^5}$, so for $1 \le x \le 3$ we have $\big|f''''(x)\big| \le 24$. Thus the error is $S_n \le \frac{2^5\cdot 24}{180\cdot 4^4} = \frac{1}{60}$.

(c) Find a value of n such that if we approximate $\displaystyle\int_0^1 \frac{4\,dx}{1+x^2}$ by M_n, the error is $E_n \le \frac{1}{300}$.

Solution: Let $f(x) = \dfrac{4}{1+x^2}$. When we approximate $\displaystyle\int_0^1 f(x)\,dx$ using M_n, the absolute error is

$$|E_n| \le \frac{1(1-0)^3}{24\,n^2}\cdot K = \frac{K}{24\,n^2}\ ,\ \text{where } K = \max_{0\le x\le 1}\big|f''(x)\big|\,.$$

Verify that

$$f'(x) = \frac{-8\,x}{(1+x^2)^2}\ ,\quad f''(x) = \frac{8(3x^2-1)}{(1+x^2)^3}\quad\text{and}\quad f'''(x) = \frac{-96\,x(x^2-1)}{(1+x^2)^4}\,.$$

Since $f'''(x) = \frac{-96x(x-1)(x+1)}{(1+x^2)^4}$, we have $f'''(x) > 0$ for $x \in (0,1)$ and so $f''(x)$ is increasing in $(0,1)$. Since $f''(x) = \frac{8(3x^2-1)}{(1+x^2)^3}$ we have $f''(0) = -8$ and $f''(1) = 2$ so $K = \max\limits_{0\le x\le 1}\big|f''(x)\big| = 8$. Thus the absolute error is

$$|E_n| \le \frac{K}{24\,n^2} = \frac{8}{24\,n^2} = \frac{1}{3\,n^2}\,.$$

To get $|E_n| \le \frac{1}{300}$ we can choose n so that $\frac{1}{3\,n^2} \le \frac{1}{300}$, that is $n^2 \ge 100$ so $n \ge 10$.

4: (a) Find the improper integral $\displaystyle\int_1^\infty \frac{dx}{x^3\sqrt{x^2-1}}$.

Solution: Make the substitution $\sec\theta = x$ so $\tan\theta = \sqrt{x^2-1}$ and $\sec\theta\tan\theta\,d\theta = dx$. Note that as $x\to\infty$ we have $\theta\to\frac{\pi}{2}$ and so

$$\begin{aligned}\int_1^\infty \frac{dx}{x^3\sqrt{x^2-1}} &= \int_0^{\pi/2}\frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta} = \int_0^{\pi/2}\frac{d\theta}{\sec^2\theta} = \int_0^{\pi/2}\cos^2\theta\,d\theta\\ &= \int_0^{\pi/2}\tfrac12+\tfrac12\cos 2\theta\;d\theta = \Big[\tfrac12\theta+\tfrac14\sin 2\theta\Big]_0^{\pi/2} = \tfrac{\pi}{4}\,.\end{aligned}$$

(b) Find the improper integral $\displaystyle\int_0^\infty \frac{e^x+1}{e^{2x}+1}\,dx$.

Solution: Make the substitution $u=e^x$ so $du = e^x\,dx$. Note that as $x\to\infty$ we have $u\to\infty$, so

$$\int_0^\infty \frac{e^x+1}{e^{2x}+1}\,dx = \int_0^\infty \frac{e^x+1}{e^x(e^{2x}+1)}\,e^x\,dx = \int_1^\infty \frac{u+1}{u(u^2+1)}\,du\,.$$

To get $\dfrac{A}{u}+\dfrac{Bu+C}{x^2+1} = \dfrac{u+1}{u(u^2+1)}$ we need $A(u^2+1)+(Bu+C)u = u+1$. Equate coefficients to get $A+B=0$, $C=1$ and $A=1$, so we have $A=1$, $B=-1$ and $C=1$, and so

$$\begin{aligned}\int_0^\infty \frac{e^x+1}{e^{2x}+1}\,dx &= \int_1^\infty \frac{u+1}{u(u^2+1)}\,du = \int_1^\infty \frac1u - \frac{x}{u^2+1}+\frac{1}{u^2+1}\;du = \Big[\ln u - \tfrac12\ln(u^2+1)+\tan^{-1}u\Big]_1^\infty\\ &= \Big[\tfrac12\ln\frac{u^2}{u^2+1}+\tan^{-1}u\Big]_1^\infty = \big(\tfrac{\pi}{2}\big)-\big(\tfrac12\ln\tfrac12+\tfrac{\pi}{4}\big) = \tfrac{\pi}{4}+\tfrac12\ln 2\,,\end{aligned}$$

since $\lim\limits_{u\to\infty}\tan^{-1}u = \frac{\pi}{2}$ and as $u\to\infty$ we have $\frac{u^2}{u^2+1}\to 1$ and so $\lim\limits_{u\to\infty}\frac12\ln\frac{u^2}{u^2+1} = \frac12\ln 1 = 0$.

5: (a) Find the improper integral $\displaystyle\int_3^\infty \frac{dx}{(x^2-1)\sqrt{x}}$.

Solution: Make the substitution $u=\sqrt{x}$ so $u^2=x$ and $2u\,du=dx$ to get

$$\int_3^\infty \frac{dx}{(x^2-1)\sqrt{x}}=\int_{\sqrt3}^\infty \frac{2u\,du}{(u^4-1)\,u}=\int_{\sqrt3}^\infty \frac{2\,du}{u^4-1}\,.$$

Note that $u^4-1=(u-1)(u+1)(u^2+1)$. To get $\dfrac{A}{u-1}+\dfrac{B}{u+1}+\dfrac{Cx+D}{u^2+1}=\dfrac{2}{u^4-1}$ we need $A(u+1)(u^2+1)+B(u-1)(u^2+1)+C(u-1)(u+1)=2$. Equate coefficients to get the 4 equations $A+B+C=0$ (1), $A-B+D=0$ (2), $A+B-C=0$ (3), and $A-B-D=2$ (4). Subtracting (3) from (1) gives $2C=0$ so $C=0$. Subtracting (4) from (2) gives $2D=-2$ so $D=-1$. Put $C=0$ into (1) to get $A+B=0$ (5), and put $D=-1$ into (2) to get $A-B=1$ (6). Adding (5) and (6) gives $2A=1$ so $A=\frac12$, and subtracting (6) from (5) gives $2B=-1$ so $B=-\frac12$. Thus we have

$$\begin{aligned}\int_3^\infty \frac{dx}{(x^2-1)\sqrt{x}}&=\int_{\sqrt3}^\infty \frac{2\,du}{u^4-1}=\int_{\sqrt3}^\infty \frac{\frac12}{u-1}-\frac{\frac12}{u+1}-\frac{1}{u^2+1}\,du\\&=\left[\tfrac12\ln\frac{u-1}{u+1}-\tan^{-1}u\right]_{\sqrt3}^\infty=\left(-\tfrac{\pi}{2}\right)-\left(\tfrac12\ln\tfrac{\sqrt3-1}{\sqrt3+1}-\tfrac{\pi}{3}\right)=\tfrac12\ln(2+\sqrt3)-\tfrac{\pi}{6}\end{aligned}$$

since $\displaystyle\lim_{u\to\infty}\tfrac12\ln\frac{u-1}{u+1}=0$ and $\displaystyle\lim_{u\to\infty}\tan^{-1}u=\tfrac{\pi}{2}$, and also $-\ln\frac{\sqrt3-1}{\sqrt3+1}=\ln\left(\frac{\sqrt3+1}{\sqrt3-1}\frac{\sqrt3+1}{\sqrt3+1}\right)=\ln(2+\sqrt3)$.

(b) Find the improper integral $\displaystyle\int_2^{17/4}\sqrt{\frac{x+2}{x-2}}\,dx$.

Solution: First we make the substitution $u=\sqrt{x-2}$ so $u^2=x-2$ and $2u\,du=dx$, and then we make the substitution $2\tan\theta=u$ so $2\sec\theta=\sqrt{u^2+4}$ and $2\sec^2\theta\,d\theta=du$. We obtain

$$\begin{aligned}\int_2^{17/4}\sqrt{\frac{x+2}{x-2}}\,dx&=\int_0^{3/2}\frac{\sqrt{u^2+4}}{u}2u\,du=\int_0^{3/2}2\sqrt{u^2+4}\,du=\int_0^{\tan^{-1}(3/4)}2\cdot2\sec\theta\cdot2\sec^2\theta\,d\theta\\&=\int_0^{\tan^{-1}(3/4)}8\sec^3\theta\,d\theta=\Big[4\sec\theta\tan\theta+4\ln(\sec\theta+\tan\theta)\Big]_0^{\tan^{-1}(3/4)}\\&=4\cdot\tfrac54\cdot\tfrac34+4\ln\left(\tfrac54+\tfrac34\right)=\tfrac{15}{4}+4\ln2\,.\end{aligned}$$

We used the fact that $\displaystyle\int\sec^3\theta\,d\theta=\tfrac12\sec\theta\tan\theta+\tfrac12\ln\big|\sec\theta+\tan\theta\big|+c$, which is shown in example 8 on page 464 in the text, and also that $\sec\left(\tan^{-1}\frac34\right)=\frac54$, which can be seen from a right-angled triangle with sides of lengths 3, 4 and 5.