MATH 138 Calculus 2, Solutions to Assignment 4

1: Let R be the region $0 \le x \le 3$, $0 \le y \le 3 + 2x - x^2$, and let S be the solid obtained by revolving R about the y-axis.

(a) Find the volume of S by integrating with respect to x.

Solution: Using the method of cylindrical shells, the volume is

$$\begin{aligned} V &= \int_0^3 2\pi x(3+2x-x^2)\,dx = \pi \int_0^3 6x + 4x^2 - 2x^3\;dx \\ &= \pi\Big[3x^2 + \tfrac{4}{3}x^3 - \tfrac{1}{2}x^4\Big]_0^3 = \pi\left(27 + 36 - \tfrac{81}{2}\right) = \tfrac{45\pi}{2}\,. \end{aligned}$$

(b) Find the volume of S by integrating with respect to y.

Solution: Note first that

$$y = 3 + 2x - x^2 \iff y = 4 - (x-1)^2 \iff (x-1)^2 = 4 - y \iff x = 1 \pm \sqrt{4-y}$$

(this formula for x in terms of y can also be obtained using the quadratic formula). The region R can be cut into two regions, one given by $0 \le y \le 3$, $0 \le x \le 1 + \sqrt{4-y}$, and the other given by $3 \le y \le 4$, $1 - \sqrt{4-y} \le x \le 1 + \sqrt{4-y}$. Using the method of cross-sections, the volume of S is given by

$$\begin{aligned} V &= \int_{y=0}^3 \pi(1+\sqrt{4-y})^2\,dy + \int_{y=3}^4 \pi\big((1+\sqrt{4-y})^2 - (1-\sqrt{4-y})^2\big)\,dy \\ &= \pi\left(\int_0^3 5 - y + 2\sqrt{4-y}\;dy + \int_3^4 4\sqrt{4-y}\;dy\right) \\ &= \pi\left(\Big[5y - \tfrac{1}{2}y^2 - \tfrac{4}{3}(4-y)^{3/2}\Big]_0^3 + \Big[-\tfrac{8}{3}(4-y)^{3/2}\Big]_3^4\right) \\ &= \pi\Big(\left(15 - \tfrac{9}{2} - \tfrac{4}{3}\right) - \left(-\tfrac{32}{3}\right) + (0) - \left(-\tfrac{8}{3}\right)\Big) \\ &= \pi\left(15 - \tfrac{9}{2} + 12\right) = \tfrac{45\pi}{2}\,. \end{aligned}$$

2: Let R be the region $1 \le x \le 2$, $0 \le y \le \dfrac{1}{x\sqrt{x^2+2x}}$.

(a) Find the volume of the solid obtained by revolving R about the x-axis.

Solution: Using cross-sections, the volume is

$$V = \int_{x=1}^{2} \pi \left(\frac{1}{x\sqrt{x^2+2x}} \right)^2 dx = \pi \int_1^2 \frac{dx}{x^2(x^2+2x)} = \pi \int_1^2 \frac{dx}{x^3(x+2)} \,.$$

To get $\dfrac{1}{x^3(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+2}$ we need $Ax^2(x+2) + Bx(x+2) + C(x+2) + Dx^3 = 1$. Equate coefficients to get $A + D = 0$, $2A + B = 0$, $2B + C = 0$ and $2C = 1$. Solve these to get $C = \frac{1}{2}$, $B = -\frac{1}{4}$, $A = \frac{1}{8}$ and $D = -\frac{1}{8}$. Thus we have

$$\begin{aligned} V &= \pi \int_1^2 \frac{\frac{1}{8}}{x} - \frac{\frac{1}{4}}{x^2} + \frac{\frac{1}{2}}{x^3} - \frac{\frac{1}{8}}{x+2}\, dx = \pi \left[\tfrac{1}{8} \ln x + \frac{1}{4x} - \frac{1}{4x^2} - \tfrac{1}{8} \ln(x+2) \right]_1^2 \\ &= \pi \Big(\left(\tfrac{1}{8} \ln 2 + \tfrac{1}{8} - \tfrac{1}{16} - \tfrac{1}{8} \ln 4 \right) - \left(\tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{8} \ln 3 \right) \Big) = \pi \left(\tfrac{1}{16} + \tfrac{1}{8} \ln \tfrac{3}{2} \right) \,. \end{aligned}$$

(b) Find the volume of the solid obtained by revolving R about the y-axis.

Solution: Using cylindrical shells, the volume is

$$V = \int_{x=1}^{2} 2\pi\, x \left(\frac{1}{x\sqrt{x^2+2x}} \right) dx = \int_1^2 \frac{2\pi\, dx}{\sqrt{x^2+2x}} \,.$$

Note that $x^2 + 2x = (x+1)^2 - 1$. Make the substitution $\sec\theta = x + 1$, $\tan\theta = \sqrt{x^2+2x}$, $\sec\theta\tan\theta\, d\theta = dx$ to get

$$\begin{aligned} \int \frac{2\pi\, dx}{\sqrt{x^2+2x}} &= \int \frac{2\pi\, \sec\theta\tan\theta\, d\theta}{\tan\theta} = \int 2\pi \sec\theta\, d\theta \\ &= 2\pi \ln \left| \sec\theta + \tan\theta \right| + c = 2\pi \ln \left| (x+1) + \sqrt{x^2+2x} \right| + c \end{aligned}$$

and so

$$V = \int_1^2 \frac{2\pi\, dx}{\sqrt{x^2+2x}} = 2\pi \Big[\ln \left((x+1) + \sqrt{x^2+2x} \right) \Big]_1^2 = 2\pi \big(\ln(3+\sqrt{8}) - \ln(2+\sqrt{3}) \big) \,.$$

3: Let R be the (infinitely long) region $0 \le x < \infty$, $0 \le y \le \dfrac{2\sqrt{x}}{4+x^2}$.

(a) Find the volume of the solid obtained by revolving R about the x-axis.

Solution: Using cross-sections, the volume is

$$V = \int_{x=0}^{\infty} \pi \left(\frac{2\sqrt{x}}{4+x^2} \right)^2 dx = \int_0^{\infty} \frac{4\pi\, x}{(4+x^2)^2}\, dx\,.$$

Make the substitution $2\tan\theta = x$, $2\sec\theta = \sqrt{4+x^2}$, $2\sec\theta\tan\theta\, d\theta$ to get

$$V = \int_{\theta=0}^{\pi/2} \frac{4\pi \cdot 2\tan\theta \cdot 2\sec^2\theta\, d\theta}{16\sec^4\theta} = \pi \int_0^{\pi/2} \frac{\tan\theta}{\sec^2\theta}\, d\theta = \pi \int_0^{\pi/2} \frac{\sec\theta\tan\theta\, d\theta}{\sec^3\theta}$$

Make the substitution $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$ to get

$$V = \pi \int_{u=1}^{\infty} \frac{du}{u^3} = \pi \left[\frac{-1}{2u^2} \right]_1^{\infty} = \tfrac{\pi}{2}\,.$$

(b) Find the area of R.

Solution: The area is

$$A = \int_0^{\infty} \frac{2\sqrt{x}}{4+x^2}\, dx\,.$$

Make the substitution $u = \sqrt{x}$, $u^2 = x$, $2u\, du = dx$ to get

$$\begin{aligned}
A &= \int_{u=0}^{\infty} \frac{2u \cdot 2u\, du}{4+u^4} = \int_0^{\infty} \frac{4u^2\, du}{(u^2-2u+2)(u^2+2u+2)} \\
&= \int_0^{\infty} \frac{u}{u^2-2u+2} - \frac{u}{u^2+2u+2}\, du = \int_0^{\infty} \frac{(u-1)+1}{u^2-2u+2} - \frac{(u+1)-1}{u^2+2u+2}\, du \\
&= \int_0^{\infty} \frac{u-1}{u^2-2u+2} + \frac{1}{u^2-2u+2} - \frac{u+1}{u^2+2u+2} + \frac{1}{u^2+2u+2}\, du \\
&= \left[\tfrac{1}{2}\ln(u^2-2u+2) + \tan^{-1}(u-1) - \tfrac{1}{2}\ln(u^2+2u+2) + \tan^{-1}(u+1) \right]_0^{\infty} \\
&= \left[\tfrac{1}{2}\ln\left(\frac{u^2-2u+2}{u^2+2u+2} \right) + \tan^{-1}(u-1) + \tan^{-1}(u+1) \right]_0^{\infty} \\
&= \left(0 + \tfrac{\pi}{2} + \tfrac{\pi}{2}\right) - \left(0 - \tfrac{\pi}{4} + \tfrac{\pi}{4}\right) = \pi\,.
\end{aligned}$$

4: Let S be the solid $0 \le x \le 2$, $-x \le y \le x$, $0 \le z \le x^2 - y^2$.

(a) Find the volume of S by integrating with respect to x.

Solution: The area of the cross-section at x is

$$A(x) = \int_{y=-x}^{x} x^2 - y^2 \; dy = \Big[x^2 y - \tfrac{1}{3} y^3\Big]_{y=-x}^{x} = \big(x^3 - \tfrac{1}{3}x^3\big) - \big(-x^3 + \tfrac{1}{3}x^3\big) = \tfrac{4}{3}\, x^3 \,,$$

and so the volume is

$$V = \int_{x=0}^{2} A(x)\, dx = \int_0^2 \tfrac{4}{3} x^3 \, dx = \Big[\tfrac{1}{3} x^4\Big]_0^2 = \tfrac{16}{3} \,.$$

(b) Find the volume of S by integrating with respect to y.

Solution: The area of the cross-section at y is

$$A(y) = \int_{x=|y|}^{2} x^2 - y^2 \; dx = \Big[\tfrac{1}{3} x^3 - y^2 x\Big]_{x=|y|}^{2} = \big(\tfrac{8}{3} - 2y^2\big) - \big(\tfrac{1}{3} y^2 |y| - y^2 |y|\big) = \tfrac{8}{3} - 2y^2 + \tfrac{2}{3} y^2 |y| \,,$$

and so the volume is

$$\begin{aligned} V &= \int_{y=-2}^{2} \tfrac{8}{3} - 2y^2 + \tfrac{2}{3} y^2 |y| \; dy = \int_{y=-2}^{0} \tfrac{8}{3} - 2y^2 - \tfrac{2}{3} y^3 \; dy + \int_0^2 \tfrac{8}{3} - 2y^2 + \tfrac{2}{3} y^3 \; dy \\ &= \Big[\tfrac{8}{3} y - \tfrac{2}{3} y^3 - \tfrac{1}{6} y^4\Big]_{-2}^{0} + \Big[\tfrac{8}{3} - \tfrac{2}{3} y^3 + \tfrac{1}{6} y^4\Big]_0^2 = -\big(-\tfrac{16}{3} + \tfrac{16}{3} - \tfrac{8}{3}\big) + \big(\tfrac{16}{3} - \tfrac{16}{3} + \tfrac{8}{3}\big) = \tfrac{16}{3} \,. \end{aligned}$$