# MATH 138 Calculus 2, Solutions to Assignment 5

**1:** (a) Verify that $y = x\sin x$ is a solution of the DE $y\,(y'' + y) = x\sin 2x$.

Solution: We have $y' = \sin x + x\cos x$ and $y'' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$ and so

$$\begin{aligned} y\,(y''+y) &= (x\sin x)(2\cos x - x\sin x + x\sin x) \\ &= (x\sin x)(2\cos x) \\ &= x(2\sin x\cos x) \\ &= x\sin 2x\,. \end{aligned}$$

(b) Find all the solutions of the form $y = ax^2 + bx + c$ to the DE $\big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1$.

Solution: For $y = ax^2 + bx + c$ we have $y' = 2ax + b$, so

$$\begin{aligned} \big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1 &\iff \big(y'(x)\big)^2 + 4x - 3y(x) - x^2 - 1 = 0 \\ &\iff (2ax+b)^2 + 4x - 3(ax^2+bx+c) - x^2 - 1 = 0 \\ &\iff (4a^2 - 3a - 1)x^2 + (4ab + 4 - 3b)x + (b^2 - 3c - 1) = 0 \\ &\iff 4a^2 - 3a - 1 = 0\ ,\ 4ab + 4 = 3b\ ,\ \text{and } b^2 = 3c + 1 \end{aligned}$$

From $4a^2 - 3a - 1 = 0$ we get $(4a+1)(a-1) = 0$ and so $a = -\frac{1}{4}$ or $a = 1$. When $= -\frac{1}{4}$, the equation $4ab + 4 = 3b$ gives $-1 + 4 = 3b$ so $b = 1$, and then the equation $b^2 = 3c + 1$ gives $1 = 3c + 1$ so $c = 0$. When $a = 1$, $4ab + 4 = 3b$ gives $4b + 4 = 3b$ so $b = -4$ and then $b^2 = 3c + 1$ gives $16 = 3c + 1$ so $c = 5$. Thus there are two solutions, and they are $y = -\frac{1}{4}x^2 + x$ and $y = x^2 - 4x + 5$.

(c) Find constants $r_1$ and $r_2$ such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\,e^{r_1 x} + b\,e^{r_2 x}$ is a solution for any constants $a$ and $b$, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

Solution: Let $y = e^{rx}$. Then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$ and so $y'' + 3y' + 2y = 0 \iff r^2\,e^{rx} + 3r\,e^{rx} + 2\,e^{rx} = 0$ $\iff (r^2 + 3r + 2)e^{rx} = 0 \iff (r+1)(r+2)\,e^{rx} = 0 \iff r = -1$ or $r = -2$. Thus we can take $r_1 = -1$ and $r_2 = -2$.

Now, let $y = a\,e^{r_1 x} + b\,e^{r_2 x} = a\,e^{-x} + b\,e^{-2x}$. Then $y' = -a\,e^{-x} - 2b\,e^{-2x}$ and $y'' = a\,e^{-x} + 4b\,e^{-2x}$ and so we have $y'' + 3y' + 2y = a\,e^{-x} + 4b\,e^{-2x} - 3a\,e^{-x} - 6b\,e^{-2x} + 2a\,e^{-x} + 2b\,e^{-2x} = 0$. This shows that $y = a\,e^{-x} + b\,e^{-2x}$ is a solution to the DE. Also, note that $y(0) = a + b$ and $y'(0) = -a - 2b$, and so to get $y(0) = 1$ and $y'(0) = 0$ we need $a + b = 1$ and $-a - 2b = 0$. Solve these two equations to get $a = 2$ and $b = -1$. Thus the required solution is $y = 2\,e^{-x} - e^{-2x}$.

**2:** Find the general solution to each of the following DEs.

(a) $x\,y' + y = \sqrt{x}$

Solution: This DE is linear since we can write it in the form $y' + \frac{1}{x}\,y = x^{-1/2}$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$ and so the solution is $y = \frac{1}{x}\int x \cdot x^{-1/2}\,dx = \frac{1}{x}\int x^{1/2}\,dx = \frac{1}{x}\left(\frac{2}{3}\,x^{3/2} + c\right) = \frac{2}{3}\sqrt{x} + \frac{c}{x}$.

(b) $\sqrt{x}\,y' = 1 + y^2$

Solution: This DE is separable. We can write it as $\dfrac{dy}{1+y^2} = x^{-1/2}\,dx$ and then integrate both sides to get $\tan^{-1} y = 2x^{1/2} + c$, that is $y = \tan\left(2\sqrt{x} + c\right)$.

(c) $y' = 2xy^2 + y^2 + 8x + 4$

Solution: This DE is separable since we can write it as $y' = (2x+1)(y^2+4)$ or as $\dfrac{dy}{y^2+4} = (2x+1)dx$. Integrate both sides to get

$$\begin{aligned} \int \frac{dy}{y^2+4} &= \int 2x + 1\ dx \\ \tfrac{1}{2}\tan^{-1}(y/2) &= x^2 + x + c \\ y &= 2\tan\left(2(x^2+x+c)\right). \end{aligned}$$

(d) $y' + y\tan x = \sin^2 x$

Solution: This DE is linear. An integrating factor is $\lambda = e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x = \dfrac{1}{\cos x}$ and the solution is

$$\begin{aligned} y &= \cos x \int \frac{\sin^2 x}{\cos x}\,dx = \cos x \int \frac{1-\cos^2 x}{\cos x}\,dx = \cos x \int \sec x - \cos x\ dx \\ &= \cos x\Big(\ln\left|\sec x + \tan x\right| - \sin x + c\Big). \end{aligned}$$

**3:** Find the solution to each of the following IVPs.

(a) $x\,y' = y^2 + y$ with $y(1) = 1$.

Solution: This DE is separable. We write it as $\dfrac{dy}{y^2+y} = \dfrac{dx}{x}$. Integrate both sides, using partial fractions for the integral on the left, to get

$$\begin{aligned}\int \frac{1}{y} - \frac{1}{y+1}\,dy &= \int \frac{1}{x}\,dx \\ \ln y - \ln(y+1) &= \ln x + c \\ \ln\left(\frac{y}{y+1}\right) &= \ln x + c \\ \frac{y}{y+1} &= e^{\ln x + c} = a\,x\,,\end{aligned}$$

where $a = \ln c$. Put in $y(1) = 1$ to get $\frac{1}{2}$, so we have $\dfrac{y}{y+1} = \dfrac{x}{2}$ so $2y = x(y+1) = xy + x$, that is $y(2-x) = x$, so the solution is $y = \dfrac{x}{2-x}$.

(b) $x\,y' + 2y = \ln x$ with $y(1) = 0$.

Solution: This DE is linear since we can write it as $y' + \frac{2}{x}\,y = \frac{1}{x}\,\ln x$. An integrating factor is given by $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and so the solution is $y = \dfrac{1}{x^2}\displaystyle\int x\,\ln x\,dx$. We integrate by parts using $u = \ln x$ and $dv = x\,dx$ so that $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}\,x^2$ to get

$$\begin{aligned}y &= \frac{1}{x^2}\int x\,\ln x\,dx \\ &= \frac{1}{x^2}\left(\tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}\,x\,dx\right) \\ &= \frac{1}{x^2}\left(\tfrac{1}{2}\,x^2\ln x - \tfrac{1}{4}\,x^2 + c\right) \\ &= \frac{c}{x^2} + \tfrac{1}{2}\,\ln x - \tfrac{1}{4}\end{aligned}$$

Put in $y(1) = 0$ to get $0 = c - \frac{1}{4}$, so we have $c = \frac{1}{4}$ and the solution is $y = \frac{1}{4}\left(\dfrac{1}{x^2} + 2\ln x - 1\right)$.

(c) $y' + xy = x^3$ with $y(0) = 1$.

Solution: This DE is linear. An integrating factor is $\lambda = e^{\int x\,dx} = e^{\frac{1}{2}x^2}$. The solution to the DE is

$$y = e^{-\frac{1}{2}x^2}\int x^3 e^{\frac{1}{2}x^2}\,dx\,.$$

Integrate by parts using $u = x^2$, $du = 2x\,dx$, $v = e^{\frac{1}{2}x^2}$, $dv = xe^{\frac{1}{2}x^2}$ to get

$$y = e^{-\frac{1}{2}x^2}\left(x^2 e^{\frac{1}{2}x^2} - \int 2xe^{\frac{1}{2}x^2}\,dx\right) = e^{-\frac{1}{2}x^2}\left(x^2e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2} + c\right) = x^2 - 2 + ce^{-\frac{1}{2}x^2}\,.$$

To get $y(0) = 1$ we need $-2 + c = 1$ so $c = 3$. Thus the solution to the IVP is

$$y = x^2 - 2 + 3e^{-\frac{1}{2}x^2} \text{ for all } x\,.$$

**4:** Solve the initial value problem $y'' - 2y' = 4x$ with $y(0) = 0$ and $y'(0) = 0$.
(Hint: first let $u(x) = y'(x)$ so that $y''(x) = u'(x)$ and then solve the resulting DE for $u = u(x)$).

Solution: When we let $u = y'$ so that $u' = y''$, the DE becomes $u' - 2u = 4x$, which is linear. An integrating factor is $\lambda = e^{\int -2\,dx} = e^{-2x}$ and so the solution is $y' = u = e^{2x}\int 4x\,e^{-2x}\,dx$. We integrate by parts using $u = 4x$ and $dv = e^{-2x}\,dx$ so that $du = 4\,dx$ and $v = -\frac{1}{2}\,e^{-2x}$ to get

$$y' = e^{2x}\int 4x\,e^{-2x}\,dx = e^{2x}\left(-2x\,e^{2x} + \int 2\,e^{-2x}\,dx\right) = e^{2x}\left(-2x\,e^{-2x} - e^{-2x} + c_1\right) = c_1\,e^{2x} - 2x - 1\,.$$

Put in $y'(0) = 0$ to get $0 = c_1 - 1$ so that $c_1 = 1$, and so we have $y' = e^{2x} - 2x - 1$. Now integrate again to get

$$y = \int e^{2x} - 2x - 1\;dx = \tfrac{1}{2}\,e^{2x} - x^2 - x + c_2\,.$$

Put in $y(0) = 0$ to get $0 = \frac{1}{2} + c_2$, so we have $c_2 = -\frac{1}{2}$, and the solution is $y = \frac{1}{2}\,e^{2x} - x^2 - x - \frac{1}{2}$.

(b) Solve the IVP $y\,y'' + (y')^2 = 0$ with $y(1) = 2$ and $y'(1) = 3$.
(Hint: first let $u(y(x)) = y'(x)$ so that $u'(y(x))y'(x) = y''(x)$ and then solve the resulting DE for $u = u(y)$).

Solution: Make the substitution $y' = u$, $y'' = u\,u'$. The DE becomes $yu\,u' + u^2 = 0$. This is linear since we can write it as $u' + \frac{1}{y}u = 0$. An integrating factor is $\lambda = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$ and the solution is $u = \frac{1}{y}\int 0\,dy = \frac{a}{y}$. Put in $x = 1$, $y = 2$, $u = y' = 3$ to get $3 = \frac{a}{2}$ so $a = 6$ and the solution is $u = \frac{6}{y}$, that is $y' = \frac{6}{y}$. This DE is separable since we can write it as $y\,y' = 6$. Integrate both sides (with respect to $x$) to get $\frac{1}{2}y^2 = 6x + c$. Put in $x = 1$, $y = 2$ to get $2 = 6 + x$ so $c = -4$ and the solution is $\frac{1}{2}y^2 = 6x - 4$, that is $y = \pm\sqrt{12x - 8}$. Since $y(1) = 2$, we must use the $+$ sign, so $y = \sqrt{12x - 8}$.