MATH 138 Calculus 2, Solutions to Assignment 6

1: (a) Find the length of the curve $y=\sqrt{4x-x^2}$ with $0\le x\le 3$.

Solution: We have $y'=\dfrac{2-x}{\sqrt{4x-x^2}}$ so the arclength is given by

$$L=\int_0^3\sqrt{1+\left(\frac{2-x}{\sqrt{4x-x^2}}\right)^2}\,dx=\int_0^3\sqrt{1+\frac{4-4x+x^2}{4x-x^2}}\,dx=\int_0^3\sqrt{\frac{4}{4x-x^2}}\,dx=\int_0^3\frac{2\,dx}{\sqrt{4x-x^2}}\,.$$

Note that $4x-x^2=4-(x-2)^2$. Let $2\sin\theta=x-2$ so that $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$L=\int_0^3\frac{2\,dx}{\sqrt{4x-x^2}}=\int_{-\pi/2}^{\pi/6}\frac{4\cos\theta\,d\theta}{2\cos\theta}=\int_{-\pi/2}^{\pi/6}2\,d\theta=\Big[2\theta\Big]_{-\pi/2}^{\pi/6}=\tfrac{\pi}{3}+\pi=\tfrac{4\pi}{3}\,.$$

(b) Find the length of the curve $y=3\,x^{2/3}$ with $0\le x\le 8$.

Solution: We have $y'=2\,x^{-1/3}$, so the arclength is given by

$$L=\int_0^8\sqrt{1+(2x^{-1/3})^2}\,dx=\int_0^8\sqrt{1+\frac{4}{x^{2/3}}}\,dx=\int_0^8\frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,dx\,.$$

Let $u=x^{2/3}+4$ so that $du=\frac{2}{3}\,x^{-1/3}\,dx$, that is $\frac{3}{2}\,du=\dfrac{1}{x^{1/3}}\,dx$. Then we have

$$L=\int_0^8\frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,dx=\int_4^8\sqrt{u}\cdot\tfrac{3}{2}\,du=\int_4^8\tfrac{3}{2}\,u^{1/2}\,du=\Big[u^{3/2}\Big]_4^8=16\sqrt{2}-8\,.$$

2: (a) Find the area of the surface which is obtained by revolving the curve $y=\sqrt{x}$ with $0\le x\le 2$ about the x-axis.

Solution: We have $y'=\dfrac{1}{2\sqrt{x}}$ so the surface area is

$$A=\int_0^2 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx=\int_0^2\pi\sqrt{4x+1}\,dx=\Big[\tfrac{\pi}{6}\,(4x+1)^{3/2}\Big]_0^2=\tfrac{\pi}{6}\,(27-1)=\tfrac{13\pi}{3}\,.$$

(If you cannot solve the integral $\displaystyle\int\pi\sqrt{4x+1}\,dx$ by inspection, then try the substitution $u=4x+1$).

(b) Find the area of the surface which is obtained by revolving the curve $y=\cos x$ with $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ about the x-axis.

Solution: We have $y'=-\sin x$. By symmetry, the surface area is given by

$$A=2\int_0^{\pi/2}2\pi\cos x\sqrt{1+\sin^2x}\,dx=\int_0^{\pi/2}4\pi\cos x\sqrt{1+\sin^2x}\,dx=\int_0^1 4\pi\sqrt{1+u^2}\,du\,,$$

where $u=\sin x$ so $du=\cos x\,dx$. Now let $\tan\theta=u$ so that $\sec\theta=\sqrt{1+u^2}$ and $\sec^2\theta\,d\theta=du$. Then

$$A=\int_0^{\pi/4}4\pi\sec^3\theta\,d\theta=\Big[2\pi\big(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\big)\Big]_0^{\pi/4}=2\pi\big(\sqrt{2}+\ln(\sqrt{2}+1)\big)\,.$$

(We used the fact that $\displaystyle\int\sec^3x\;dx=\tfrac{1}{2}\sec x\tan x+\tfrac{1}{2}\ln\big|\sec x+\tan x\big|+c$, as shown in example 2.21 in the lecture notes).

3: Consider the IVP $y' = 2(x+y) - \frac{1}{2}$ with $y(0) = 0$.

(a) Find the exact solution $y = f(x)$ to the above IVP.

Solution: The DE is linear as we can write it as $y' - 2y = 2x - \frac{1}{2}$. An integrating factor is $\lambda = e^{\int -2\,dx} = e^{-2x}$, and the solution is $y = e^{2x} \displaystyle\int \left(2x - \frac{1}{2}\right) e^{-2x}\,dx$. Integrate by parts using $u = 2x - \frac{1}{2}$, $du = 2\,dx$, $v = \frac{-1}{2} e^{-2x}$ and $dv = e^{-2x}\,dx$ to get

$$y = e^{2x}\left(\left(-x + \tfrac{1}{4}\right) e^{-2x} + \int e^{-2x}\,dx\right) = e^{2x}\left(\left(-x + \tfrac{1}{4}\right) e^{-2x} - \tfrac{1}{2} e^{-2x} + c\right) = ce^{2x} - \left(x + \tfrac{1}{4}\right).$$

To get $y(0) = 0$ we need $0 = c - \frac{1}{4}$ so $c = \frac{1}{4}$, and the solution is

$$y = \tfrac{1}{4} e^{2x} - \left(x + \tfrac{1}{4}\right).$$

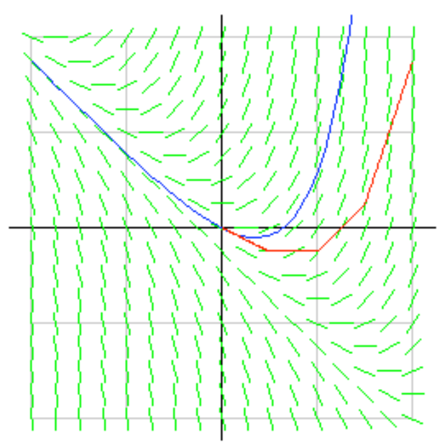
(b) Apply Euler's method with step size $\Delta x = \frac{1}{2}$ to find a polygonal approximation $y = g(x)$ for $0 \le x \le 2$ to the above solution $y = f(x)$.

Solution: We make a table showing the values of x_k, y_k and m_k, where $x_0 = 0$, $y_0 = 0$, $m_k = 2(x_k + y_k) - \frac{1}{2}$, $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + m_k \Delta x$.

k	x_k	y_k	m_k
0	0	0	$2(0+0) - \frac{1}{2} = -\frac{1}{2}$
1	$\frac{1}{2}$	$0 + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$	$2\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2} = 0$
2	1	$-\frac{1}{4} + (0)\left(\frac{1}{2}\right) = -\frac{1}{4}$	$2\left(1 - \frac{1}{4}\right) - \frac{1}{2} = 1$
3	$\frac{3}{2}$	$-\frac{1}{4} + (1)\left(\frac{1}{2}\right) = \frac{1}{4}$	$2\left(\frac{3}{2} + \frac{1}{4}\right) - \frac{1}{2} = 3$
4	2	$\frac{1}{4} + (3)\left(\frac{1}{2}\right) = \frac{7}{4}$	

(c) Sketch the direction field for the given DE along with the graph of the exact solution $y = f(x)$ and the graph of the polygonal solution $y = g(x)$.

Solution: The direction field is shown in green, the exact solution is in blue, and the polygonal approximation is in red.



4: (a) The amount $A(t)$ of a radioactive substance satisfies the DE

$$A'(t) = k\,A(t)$$

for some constant $k<0$. The substance has a half-life of 10 seconds, which means that $A(10) = \frac{1}{2}\,A(0)$. If $A(5) = 100$ then find the exact time t at which $A(t) = 20$.

Solution: This DE is linear, since we can write it in the form $A' + kA = 0$. An integrating factor is $\lambda = e^{\int k\,dt} = e^{k\,t}$ and the general solution is $A(t) = e^{-k\,t}\int 0\,dt = c\,e^{-kt}$. Note that $A(0) = c$, so c is the initial amount. Since the half-life is 10, we have

$$A(10) = \tfrac{1}{2}\,c \Longrightarrow c\,e^{-10k} = \tfrac{1}{2}\,c \Longrightarrow e^{-10k} = \tfrac{1}{2} \Longrightarrow e^{10k} = 2 \Longrightarrow 10k = \ln 2 \Longrightarrow k = \tfrac{1}{10}\ln 2\,.$$

and so $A(t) = c\,e^{-(t/10)\ln 2} = c\,2^{-t/10}$. Also, we have $A(5) = 100 \Longrightarrow c\,2^{-1/2} = 100 \Longrightarrow c = 100\sqrt{2}$, and so $A(t) = \big(100\sqrt{2}\big)\,2^{-t/10}$. Finally, we have

$$\begin{aligned}
A(t) = 20 &\iff \big(100\sqrt{2}\big)\,2^{-t/10} = 20 \\
&\iff 2^{t/10} = \frac{100\sqrt{2}}{20} = 5\sqrt{2} = \sqrt{50} \\
&\iff \frac{t}{10} = \log_2 \sqrt{50} = \tfrac{1}{2}\log_2 50 \\
&\iff t = 5\log_2 50\,.
\end{aligned}$$

(b) A murder victim is found in a room of constant teperature 25° C. At the time of murder, we assume that the victim's body temperature was 37° C. At 2:00 pm, the body temperature is measured to be 31° C and at 5:00 pm, it is measured to be 29° C. Determine the time of death, assuming that the temperature $T = T(t)$ of the body at time t satisfies Newton's Law of Cooling, so that $T' = -k(T-25)$ for some constant $k>0$.

Solution: Let t hours be the time elapsed since 2:00 pm (so $t=0$ at 2:00). The DE $T' = -k(T-25)$ is linear since we can write it as $T' + kT = 25k$. An integrating factor is $I = e^{\int k\,dt} = e^{kt}$ and the solution is

$$T = e^{-kt}\int 25k\,e^{kt}\,dt = e^{-kt}\big(25\,e^{kt} + c\big) = 25 + c\,e^{-kt}\,.$$

To get $T(0) = 31$ we need $25 + c = 31$ so $c = 6$, and so we have

$$T(t) = 25 + 6\,e^{-kt}\,.$$

Also we have $T(3) = 29 \Longrightarrow 25 + 6\,e^{-3k} = 29 \Longrightarrow 6\,e^{-3k} = 4 \Longrightarrow -3k = \frac{2}{3} \Longrightarrow k = -\frac{1}{3}\ln\frac{2}{3} = \frac{1}{3}\ln\frac{3}{2}$. Finally, $T(t) = 37 \iff 25 + 6\,e^{-kt} = 37 \iff 6\,e^{-kt} = 12 \iff e^{-kt} = 2 \iff -kt = \ln 2$ so $t = -\frac{\ln 2}{k} = -\frac{3\ln 2}{\ln\frac{3}{2}} \cong -5.13$. Thus the victim died about 5 hours before 2:00, that is at about 9:00 am.

5: A tank initially contains 20 L of pure water. Brine containing 5 grams of salt per liter of water enters the tank at 6 L/min. The solution is kept well mixed and drains from the tank at 2 L/min. Find the concentration of salt in the tank when the tank contains 80 L of brine.

Solution: Let $S(t)$ be the amount of salt in the tank (in grams) at time t (in minutes), and let $V(t)$ be the volume of brine in the tank (in litres). Note that $S(0) = 0$ and $V(0) = 20$. Also, let r_{in} and r_{out} be the incoming and outgoing rates, and let c_{in} and c_{out} be the incoming and outgoing concentrations. We have $r_{in} = 6$, $r_{out} = 2$, $c_{in} = 5$ and $c_{out} = S(t)/V(t)$.

The volume $V(t)$ satisfies the IVP $V' = r_{in} - r_{out} = 6 - 2 = 4$ with $V(0) = 20$, and the solution is easily found to be $V(t) = 20 + 4t$. The amount of salt $S(t)$ then satisfies the IVP

$$S' = r_{in}c_{in} - r_{out} - c_{out} = 6 \cdot 5 - 2\,\frac{S}{20+4t} = 30 - \frac{S}{10+2t}$$

with $S(0) = 0$. The DE is linear since we can write it as $S' + \dfrac{1}{2(5+t)}\,S = 30$. An integrating factor is $\lambda = e^{\int \frac{1}{2(5+t)}\,dt} = e^{\frac{1}{2}\ln(5+t)} = (5+t)^{1/2}$, and the general solution is

$$S(t) = (5+t)^{-1/2}\int 30\,(5+t)^{1/2}\,dt = (5+t)^{-1/2}\big(20\,(5+t)^{3/2} + c\big) = 20\,(5+t) + \frac{c}{\sqrt{5+t}}\,.$$

Since $S(0) = 0$ we have $100 + c/\sqrt{5} = 0$ and so $c = -100\sqrt{5}$ and the solution is

$$S(t) = 20\,(5+t) - \frac{100\sqrt{5}}{\sqrt{5+t}}\,.$$

The tank contains 80 litres when $V(t) = 80$, that is when $20 + 4t = 80$, so $t = 15$. At $t = 15$, the amount of salt is $S(15) = 20 \cdot 20 - \frac{100\sqrt{5}}{\sqrt{20}} = 400 - 50 = 350$, and the concentration of salt is $\dfrac{S(15)}{V(15)} = \dfrac{350}{80} = \dfrac{35}{8}$.