MATH 138 Calculus 2, Solutions to Assignment 7

1: Find the sum of each of the following series, if the sum exists.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}-1}{3^n}$

Solution: $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}-1}{3^n} = \sum_{n=0}^{\infty} 2\left(\tfrac{2}{3}\right)^n - \left(\tfrac{1}{3}\right)^n = 2\,\frac{1}{1-\frac{2}{3}} - \frac{1}{1-\frac{1}{3}} = 6 - \tfrac{3}{2} = \tfrac{9}{2}.$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{4^{1-n}}{3^{n-1}}$

Solution: $\displaystyle\sum_{n=2}^{\infty} \frac{4^{1-n}}{3^{n-1}} = \sum_{n=2}^{\infty} \frac{4\left(\frac{1}{4}\right)^n}{\frac{1}{3}\cdot 3^n} = \sum_{n=2}^{\infty} 12\left(\tfrac{1}{12}\right)^n = 12\,\frac{\left(\frac{1}{12}\right)^2}{1-\frac{1}{12}} = \tfrac{1}{11}.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+3n}$

Solution: The ℓ^{th} partial sum is

$$\begin{aligned} S_\ell &= \sum_{n=1}^{\ell} \frac{3}{n^2+3n} = \sum_{n=1}^{\ell} \left(\tfrac{1}{n} - \tfrac{1}{n+3}\right) \\ &= \left(1-\tfrac{1}{4}\right) + \left(\tfrac{1}{2}-\tfrac{1}{5}\right) + \left(\tfrac{1}{3}-\tfrac{1}{6}\right) + \left(\tfrac{1}{4}-\tfrac{1}{7}\right) + \left(\tfrac{1}{5}-\tfrac{1}{8}\right) + \cdots \\ &\quad + \left(\tfrac{1}{\ell-3}-\tfrac{1}{\ell}\right) + \left(\tfrac{1}{l\ell 2}-\tfrac{1}{\ell+1}\right) + \left(\tfrac{1}{\ell-1}-\tfrac{1}{\ell+2}\right) + \left(\tfrac{1}{\ell}-\tfrac{1}{\ell+3}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{\ell+1} - \tfrac{1}{\ell+2} - \tfrac{1}{\ell+3}\,, \end{aligned}$$

since all the other terms cancel out. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+3n} = \lim_{\ell\to\infty} S_\ell = 1 + \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{11}{6}.$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)!}$

Solution: Note that $\frac{1}{n!} - \frac{1}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{n}{(n+1)!}$ and so

$$\begin{aligned} S_\ell &= \sum_{n=0}^{\ell} \frac{n}{(n+1)!} = \sum_{n=0}^{\ell} \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) \\ &= \left(\tfrac{1}{0!}-\tfrac{1}{1!}\right) + \left(\tfrac{1}{1!}-\tfrac{1}{2!}\right) + \left(\tfrac{1}{2!}-\tfrac{1}{3!}\right) + \cdots + \left(\tfrac{1}{(\ell-1)!}-\tfrac{1}{\ell!}\right) + \left(\tfrac{1}{\ell!}-\tfrac{1}{(\ell+1)!}\right) \\ &= \tfrac{1}{0!} - \tfrac{1}{(\ell+1)!} = 1 - \tfrac{1}{(\ell+1)!} \end{aligned}$$

and so $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)!} = \lim_{\ell\to\infty} S_\ell = \lim_{l\to\infty}\left(1 - \tfrac{1}{(\ell+1)!}\right) = 1.$

2: Determine which of the following series converge.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{2n^2+1}$

Solution: For $n \geq 1$ we have $0 \leq \dfrac{\sqrt{n}}{2n^2+1} \leq \dfrac{\sqrt{n}}{2n^2} = \dfrac{1}{2n^{3/2}}$, and we know that $\sum \dfrac{1}{2n^{3/2}}$ converges (since it is a constant multiple of the p-series with $p = \frac{3}{2}$), and so $\sum \dfrac{\sqrt{n}}{2n^2+1}$ converges by the C.T.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\, 2^{1/n}$

Solution: Let $a_n = (-1)^n 2^{1/n}$. Then $|a_n| = 2^{1/n} \longrightarrow 2^0 = 1$. Since $|a_n| \not\to 0$, we know that $a_n \not\to 0$, and so $\sum a_n$ diverges by the D.T.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$ so that $a_n = f(n)$. For $x > 1$, $f(x)$ is positive, continuous and decreasing. Making the substitution $u = \ln x$ so that $du = \frac{1}{x}\,dx$, we have

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\infty} \frac{du}{\sqrt{u}} = \Big[2\sqrt{u}\Big]_{\ln 2}^{\infty} = \infty\,.$$

Since $\displaystyle\int_2^{\infty} f(x)\,dx$ diverges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges too, by the I.T.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$

Solution: Let $a_n = \dfrac{n^n}{3^n n!}$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}}{3^{n+1}(n+1)!}\,\dfrac{3^n n!}{n^n} = \dfrac{1}{3}\left(\dfrac{n+1}{n}\right)^n \longrightarrow \dfrac{e}{3} < 1$, so $\sum a_n$ converges by the R.T. (If you don't remember that $\lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$, then use l'Hôpital's Rule).

3: (a) Approximate the sum $S=\sum_{n=0}^{\infty}\frac{1}{5^n+5n}$ by a partial sum S_ℓ so that the error is $S-S_\ell\le\frac{1}{500}$.

Solution: If we approximate S by $S\cong S_\ell$, then by the C.T, the error is

$$S-S_\ell=\sum_{n=\ell+1}^{\infty}\frac{1}{5^n+5n}\le\sum_{n=\ell+1}^{\infty}\frac{1}{5^n}=\frac{\frac{1}{5^{\ell+1}}}{1-\frac15}=\frac{1}{4\cdot 5^\ell}\,.$$

To get $S-S_\ell\le\frac{1}{500}$, we want $\frac{1}{4\cdot 5^\ell}\le\frac{1}{500}$, that is $5^\ell\ge 125$, so we can take $\ell=3$. Thus we make the approximation $S\cong S_3=\sum_{n=0}^{3}\frac{1}{5^n+5n}\cong 1+\frac{1}{10}+\frac{1}{35}+\frac{1}{140}=\frac{140+14+4+1}{140}=\frac{159}{140}$

(b) Let $S=\sum_{n=1}^{\infty}\frac{n}{e^n}$. Use a calculator to find a value of ℓ such that $S-S_\ell\le\frac{1}{500}$.

Solution: Let $f(x)=x\,e^{-x}$ so that $a_n=f(n)$. Clearly $f(x)$ is positive and continuous for $x>0$. Also, $f'(x)=e^{-x}-x\,e^{-x}=(1-x)e^{-x}$ so that $f'(x)<0$ when $x>1$, and so $f(x)$ is decreasing for $x\ge 1$. By the I.T, we have

$$S-S_\ell=\sum_{n=\ell+1}^{\infty}a_n\le\int_\ell^\infty f(x)\,dx=\int_\ell^\infty x\,e^{-x}\,dx\,.$$

We integrate by parts using $u=x$ and $dv=e^{-x}\,dx$ so that $du=dx$ and $v=-e^{-x}$ to get

$$\int x\,e^{-x}\,dx=-x\,e^{-x}+\int e^{-x}\,dx=-x\,e^{-x}-e^{-x}+c=-\frac{x+1}{e^x}+c$$

and so

$$S-S_\ell\le\left[-\frac{x+1}{e^x}\right]_\ell^\infty=\frac{\ell+1}{e^\ell}$$

since $\lim_{x\to\infty}\frac{x+1}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$ by l'Hôpital's Rule. In order to get $S-S_\ell\le\frac{1}{500}$, we want $\frac{\ell+1}{e^\ell}\le\frac{1}{500}$. A calculator shows that $\frac{10}{e^9}\cong 0.0012<\frac{1}{500}$, so we can choose $\ell=9$.

(c) Let $f(x)=\frac{1}{x(\ln x)^2}$, let $a_n=f(n)$ for $n\ge 2$, let $S=\sum_{n=2}^{\infty}a_n$, and let $S_\ell=\sum_{n=2}^{\ell}a_n$. Find one value of ℓ such that if we approximate S by $S\cong S_\ell$ then the absolute error is $|S-S_\ell|\le\frac{1}{100}$ and, using a calculator, find another value of ℓ such that if we approximate S by

$$S\cong T_\ell=S_\ell+\tfrac12\left(\int_\ell^\infty f(x)\,dx+\int_{\ell+1}^\infty f(x)\,dx\right)$$

then the absolute error is $|S-T_\ell|\le\frac{1}{100}$.

Solution: Note that $f(x)$ is positive, continuous and decreasing for $x>1$, so we can apply the I.T. If we approximate S by $S\cong S_\ell$ then by the IT the error is

$$E=S-S_\ell=\sum_{n=\ell+1}^{\infty}a_n\le\int_\ell^\infty f(x)\,dx=\int_\ell^\infty\frac{dx}{x(\ln x)^2}=\left[\frac{-1}{\ln x}\right]_\ell^\infty=\frac{1}{\ln\ell}\,.$$

To get $E\le\frac{1}{100}$ we can choose ℓ so that $\frac{1}{\ln\ell}\ge\frac{1}{100}$, that is $\ln\ell\ge 100$. We can take $\ell\ge e^{100}$ (a huge number).

By the I.T, if we make the approximation $S\cong T_\ell$ then the absolute error is

$$E\le\tfrac12\left(\int_\ell^\infty f(x)\,dx-\int_{\ell+1}^\infty f(x)\,dx\right)=\tfrac12\left(\frac{1}{\ln\ell}-\frac{1}{\ln(\ell+1)}\right),$$

so to get $E\le\frac{1}{100}$ we can choose ℓ so that $\frac12\left(\frac{1}{\ln\ell}-\frac{1}{\ln(\ell+1)}\right)\le\frac{1}{100}$. By trial and error with the help of a calculator, we find that $\frac12\left(\frac{1}{\ln(10)}-\frac{1}{\ln(11)}\right)\cong 0.0086<\frac{1}{100}$ and so we can take $\ell=10$.

4: Determine, with proof, which of the following statements are true for all sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$.

(a) If $\sum a_n$ converges that $\sum e^{a_n}$ diverges.

Solution: This is true, and we give a proof. Suppose that $\sum a_n$ converges. Then $\lim\limits_{n\to\infty} a_n = 0$ by the D.T, and so $\lim\limits_{n\to\infty} e^{a_n} = e^0 = 1$. Since $\lim\limits_{n\to\infty} e^{a_n} \neq 0$, $\sum e^{a_n}$ diverges by the D.T.

(b) If $\sum a_n$ converges then $\sum {a_n}^2$ converges.

Solution: This is false, and we provide a counterexample. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum a_n$ converges by the A.S.T, but $\sum {a_n}^2 = \sum \frac{1}{n}$ which diverges.

(c) If $a_n \geq 0$ for all n and $\sum a_n$ converges then $\sum \dfrac{a_n}{1+a_n}$ converges.

Solution: This is true and we give a proof. Suppose that $\sum a_n$ converges. Note that $\lim\limits_{n\to\infty} a_n = 0$ by the D.T. Let $b_n = \dfrac{a_n}{1+a_n}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{1+a_n} = \dfrac{1}{1+\lim\limits_{n\to\infty} a_n} = \dfrac{1}{1+0} = 1$. Since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$ and $\sum a_n$ converges, $\sum b_n$ converges by the L.C.T.

(d) If $\sum a_n$ converges and $\sum |b_n|$ converges then $\sum a_n b_n$ converges.

Solution: This is true. Indeed, suppose that $\sum a_n$ converges and that $\sum |b_n|$ converges. Since $\sum a_n$ converges, we have $a_n \to 0$ (by the Divergence Test) so we can choose N so that $n \geq N \implies |a_n| \leq 1$. Then for $n \geq N$ we have $0 \leq |a_n b_n| = |a_n||b_n| \leq |b_n|$, and so, since $\sum |b_n|$ converges, $\sum |a_n b_n|$ also converges by the Comparison Test. Since absolute convergence implies convergence, $\sum a_n b_n$ converges, too.