MATH 138 Calculus 2, Solutions to Assignment 8

1: (a) Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(2-3x)^n}{n}$.

Solution: Let $a_n=\dfrac{(2-3x)^n}{n}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right|=\dfrac{|2-3x|^{n+1}}{n+1}\,\dfrac{n}{|2-3x|^n}=\dfrac{n+1}{n}\,|2-3x|\longrightarrow|2-3x|=3\left|x-\frac{2}{3}\right|$. By the R.T, the power series converges when $3\left|x-\frac{2}{3}\right|<1$, that is when $\left|x-\frac{2}{3}\right|<\frac{1}{3}$, and diverges when $\left|x-\frac{2}{3}\right|>\frac{1}{3}$, so the radius of convergence is $R=\frac{1}{3}$. The endpoints of the interval of convergence are the points where $\left|x-\frac{2}{3}\right|=\frac{1}{3}$, that is $x=\frac{1}{3}$ and $x=1$. When $x=\frac{1}{3}$, we have $a_n=\dfrac{1}{n}$ and so $\sum a_n$ diverges. When $x=1$ we have $a_n=\dfrac{(-1)^n}{n}$ and so $\sum a_n$ converges by the A.S.T. Thus the interval of convergence is $I=\left(\frac{1}{3},1\right]$.

(b) Find the Taylor series centred at 0 for $f(x)=\dfrac{x}{x^2-6x+8}$, and find its radius of convergence.

Solution: We have

$$f(x)=\frac{x}{x^2-6x+8}=\frac{x}{(x-2)(x-4)}=\frac{-1}{x-2}+\frac{2}{x-4}=\frac{\frac{1}{2}}{1-\frac{x}{2}}-\frac{\frac{1}{2}}{1-\frac{x}{4}}\,.$$

Since $\dfrac{\frac{1}{2}}{1-\frac{x}{2}}=\displaystyle\sum_{n=0}^{\infty}\tfrac{1}{2}\left(\tfrac{x}{2}\right)^n=\sum_{n=0}^{\infty}\tfrac{1}{2\cdot 2^n}x^n$ when $\left|\frac{x}{2}\right|<1$, that is $|x|<2$, and $\dfrac{\frac{1}{2}}{1-\frac{x}{4}}=\displaystyle\sum_{n=0}^{\infty}\tfrac{1}{2\cdot 4^n}x^n$ when $|x|<4$, we have

$$f(x)=\sum_{n=0}^{\infty}\tfrac{1}{2\cdot 2^n}x^n-\sum_{n=0}^{\infty}\tfrac{1}{2\cdot 4^n}x^n=\sum_{n=0}^{\infty}\tfrac{1}{2}\left(\tfrac{1}{2^n}-\tfrac{1}{4^n}\right)x^n$$

when $|x|<2$.

(c) Find the Taylor series centred at 3 for $f(x)=\dfrac{x}{x^2-6x+8}$, and find its radius of convergence.

Solution: We have

$$f(x)=\frac{x}{x^2-6x+8}=\frac{x}{(x-3)^2-1}=\frac{(x-3)+3}{(x-3)^2-1}=-\big((x-3)+3\big)\cdot\frac{1}{1-(x-3)^2}\,.$$

When $|x-3|<1$ (hence also $|x-3|^2<1$) we have $\frac{1}{1-(x-3)^2}=\sum\limits_{n=0}^{\infty}(x-3)^{2n}$, and so

$$f(x)=-\big((x-3)+3\big)\sum_{n=0}^{\infty}(x-3)^{2n}=-\sum_{n=0}^{\infty}(x-3)^{2n+1}-3\sum_{n=0}^{\infty}(x-3)^{2n}=\sum_{n=0}^{\infty}c_n(x-3)^n$$

where $c_n=-3$ when n is even and $c_n=-1$ when n is odd. The radius of convergence is 1. Equivalently, we have $c_n=(-1)^{n+1}-2$ for all n, so

$$f(x)=\sum_{n=0}^{\infty}\big((-1)^{n+1}-2\big)(x-3)^n\,.$$

2: (a) Find the Taylor polynomial of degree 4 centered at 0 for $f(x) = \dfrac{\ln(1+x)}{e^{2x}}$.

Solution: We have

$$\begin{aligned} f(x) &= e^{-2x}\ln(1+x) \\ &= \Big(1 + (-2x) + \tfrac{1}{2!}(-2x)^2 + \tfrac{1}{3!}(-2x)^3 + \tfrac{1}{4!}(-2x)^4 + \cdots\Big)\Big(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots\Big) \\ &= \Big(1 - 2x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{2}{3}x^4 - \cdots\Big)\Big(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots\Big) \\ &= x - \big(\tfrac{1}{2} + 2\big)x^2 + \big(\tfrac{1}{3} + 1 + 2\big)x^3 - \big(\tfrac{1}{4} + \tfrac{2}{3} + 1 + \tfrac{4}{3}\big)x^4 + \cdots \\ &= x - \tfrac{5}{2}x^2 + \tfrac{10}{3}x^3 - \tfrac{13}{4}x^4 + \cdots \end{aligned}$$

so the Taylor polynomial of degree 4 is $T_4(x) = x - \frac{5}{2}x^2 + \frac{10}{3}x^3 - \frac{13}{4}x^4$.

(b) Find the Taylor polynomial of degree 5 centered at 0 for $f(x) = \dfrac{(1+3x)^{4/3}}{1+x}$.

Solution: We have

$$\begin{aligned} (1+3x)^{4/3} &= 1 + \tfrac{4}{3}(3x) + \frac{\big(\frac{4}{3}\big)\big(\frac{1}{3}\big)}{2!}(3x)^2 + \frac{\big(\frac{4}{3}\big)\big(\frac{1}{3}\big)\big(-\frac{2}{3}\big)}{3!}(3x)^3 + \frac{\big(\frac{4}{3}\big)\big(\frac{1}{3}\big)\big(-\frac{2}{3}\big)\big(-\frac{5}{3}\big)}{4!}(3x)^4 \\ &\quad + \frac{\big(\frac{4}{3}\big)\big(\frac{1}{3}\big)\big(-\frac{2}{3}\big)\big(-\frac{5}{3}\big)\big(-\frac{8}{3}\big)}{5!}(3x)^5 + \cdots \\ &= 1 + 4x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{5}{3}x^4 - \tfrac{8}{3}x^5 + \cdots\,, \end{aligned}$$

and so

$$\begin{aligned} f(x) &= \frac{1}{1+x}\,(1+3x)^{4/3} \\ &= \Big(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\Big)\Big(1 + 4x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{5}{3}x^4 - \tfrac{8}{3}x^5 + \cdots\Big) \\ &= 1 + (4-1)x + (2-4+1)x^2 + \big(-\tfrac{4}{3} - 2 + 4 - 1\big)x^3 + \big(\tfrac{5}{3} + \tfrac{4}{3} + 2 - 4 + 1\big)x^4 \\ &\quad + \big(-\tfrac{8}{3} - \tfrac{5}{3} - \tfrac{4}{3} - 2 + 4 - 1\big)x^5 + \cdots \\ &= 1 + 3x - x^2 - \tfrac{1}{3}x^3 + 2x^4 - \tfrac{14}{3}x^5 + \cdots\,. \end{aligned}$$

Thus the 5^{th} Taylor polynomial is $T_5(x) = 1 + 3x - x^2 - \frac{1}{3}x^3 + 2x^4 - \frac{14}{3}x^5$.

3: (a) Approximate the value of $\ln(5/4)$ so the absolute error is at most $\frac{1}{1000}$.

Solution: $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, so we have

$$\begin{aligned}\ln\left(\tfrac{5}{4}\right) &= \tfrac{1}{4} - \tfrac{1}{2\cdot 4^2} + \tfrac{1}{3\cdot 4^3} - \tfrac{1}{4\cdot 4^4} + \cdots \\ &\cong \tfrac{1}{4} - \tfrac{1}{2\cdot 4^2} + \tfrac{1}{3\cdot 4^3} \\ &= \tfrac{1}{4} - \tfrac{1}{32} + \tfrac{1}{192} = \tfrac{43}{192}\end{aligned}$$

with error $E \leq \frac{1}{4\cdot 4^4} = \frac{1}{1024} < \frac{1}{1000}$ by the A.S.T.

(b) Approximate the value of $\sqrt[5]{e}$ so that the absolute error is at most $\frac{1}{1000}$

Solution: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$, so we have

$$\begin{aligned}\sqrt[5]{e} = e^{\frac{1}{5}} &= 1 + \tfrac{1}{5} + \tfrac{1}{5^2 2!} + \tfrac{1}{5^3 3!} + \tfrac{1}{5^4 4!} + \cdots \\ &\cong 1 + \tfrac{1}{5} + \tfrac{1}{5^2 2!} + \tfrac{1}{5^3 3!} \\ &= 1 + \tfrac{1}{5} + \tfrac{1}{50} + \tfrac{1}{750} = \tfrac{916}{750}\end{aligned}$$

with error

$$\begin{aligned}E &= \tfrac{1}{5^4 4!} + \tfrac{1}{5^5 5!} + \tfrac{1}{5^6 6!} + \cdots \\ &= \tfrac{1}{5^4 4!}\left(1 + \tfrac{1}{5\cdot 5} + \tfrac{1}{5^2\cdot 5\cdot 6} + \tfrac{1}{5^3\cdot 5\cdot 6\cdot 7} + \cdots\right) \\ &\leq \tfrac{1}{5^4 4!}\left(1 + \tfrac{1}{5^2} + \tfrac{1}{5^4} + \tfrac{1}{5^6} + \cdots\right) \\ &= \tfrac{1}{5^4 4!}\,\tfrac{1}{1-\frac{1}{25}} = \tfrac{1}{5^4 4!}\,\tfrac{25}{24} = \tfrac{1}{13200}\end{aligned}$$

where we used the C.T. and the formula for the sum of a geometric series.

(c) Approximate the value of the definite integral $\displaystyle\int_0^1 \sqrt{4+x^3}\,dx$ so the absolute error is at most $\frac{1}{1000}$.

Solution: Using the Binomial Series, we have

$$\begin{aligned}\sqrt{4+x^3}\,dx &= 2\left(1+\frac{x^3}{4}\right)^{1/2} \\ &= 2\left(1 + \tfrac{1}{2}\left(\frac{x^3}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x^3}{4}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x^3}{4}\right)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}\left(\frac{x^3}{4}\right)^4 + \cdots\right) \\ &= 2 + \tfrac{1}{4}\,x^3 - \tfrac{1}{2\cdot 2!\cdot 4^2}\,x^6 + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3}\,x^9 - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4}\,x^{12} + \cdots\end{aligned}$$

and so

$$\begin{aligned}\int_0^1 \sqrt{4+x^3}\,dx &= \left[2\,x + \tfrac{1}{4\cdot 4}\,x^4 - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7}\,x^7 + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10}\,x^{10} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4\cdot 13}\,x^{13} + \cdots\right]_0^1 \\ &= 2 + \tfrac{1}{4\cdot 4} - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7} + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4\cdot 13} + \cdots \\ &\cong 2 + \tfrac{1}{4\cdot 4} - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7} = 2 + \tfrac{1}{16} - \tfrac{1}{448} = \tfrac{923}{448}\end{aligned}$$

with absolute error $E \leq \frac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10} = \frac{1}{5120}$ by the A.S.T.

To be rigorous, we should justify our application of the A.S.T. When $a_n = \dfrac{(-1)^{n+1} 1\cdot 3\cdot 5\cdots(2n-3)}{2^{n-1}\cdot n!\cdot 4^n\cdot(3n+1)}$ we have $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{2n-1}{2\cdot(n+1)\cdot 4\cdot(3n+4)} < \dfrac{2n+2}{2\cdot(n+1)\cdot 4\cdot(3n+4)} = \dfrac{1}{4\cdot(3n+4)}$. Since $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ we know that $\{|a_n|\}$ is decreasing, and since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ we know that $\sum a_n$ converges by the R.T. so $\displaystyle\lim_{n\to\infty}|a_n| = 0$ by the D.T. Thus we can indeed apply the A.S.T.

4: (a) Find the tenth derivative $f^{(10)}(0)$, where $f(x) = \sin(x^2/2)$.

Solution: We have $f(x) = \sin\left(\frac{x^2}{2}\right) = \left(\frac{x^2}{2}\right) - \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \frac{1}{5!}\left(\frac{x^2}{2}\right)^5 - \cdots = \frac{1}{2}x^2 - \frac{1}{2^3 3!}x^6 + \frac{1}{2^5 5!}x^{10} - \cdots$ and so $f^{(10}(0) = 10!\, c_{10} = \frac{10!}{2^5 5!} = 945$.

(b) Evaluate $\displaystyle\lim_{x\to 0} \frac{\cos x - \sqrt{1-x^2}}{x(\sin x - x)}$.

Solution: We have

$$\frac{\cos x - \sqrt{1-x^2}}{x(\sin x - x)} = \frac{\left(1 - \frac{1}{2}\,x^2 + \frac{1}{24}\,x^4 - \cdots\right) - \left(1 - \frac{1}{2}\,x^2 - \frac{1}{8}\,x^4 + - \cdots\right)}{x\big(\left(x - \frac{1}{6}\,x^3 + - \cdots\right) - x\big)} = \frac{\frac{1}{6}\,x^4 + \cdots}{-\frac{1}{6}\,x^4 + \cdots} = -1 + \cdots$$

(the dots indicate there are more terms of higher order in x) and so $\displaystyle\lim_{x\to 0} \frac{\cos x - \sqrt{1-x^2}}{x(\sin x - x)} = -1$

(c) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+2)!}$.

Solution: We have

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
e^{-2} &= 1 - 2 + \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!} + \cdots \\
e^{-2} + 1 &= \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!} + \cdots \\
\frac{e^{-2} + 1}{4} &= \frac{1}{2!} + \frac{(-2)^1}{3!} + \frac{(-2)^2}{4!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+2)!}
\end{aligned}$$

(d) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{3^n}$.

Solution: For $|x| < 1$ we have

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \cdots \\
\frac{-1}{(1+x)^2} &= \frac{d}{dx}\left(\frac{1}{1+x}\right) = -1 + 2x - 3x^2 + 4x^3 - \cdots \\
\frac{-x}{(1+x)^2} &= -x + 2x^2 - 3x^3 + 4x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n n\, x^n\,.
\end{aligned}$$

Put in $x = \frac{1}{3}$ to get $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-\frac{1}{3}}{\left(1 + \frac{1}{3}\right)^2} = -\frac{3}{16}$.