Chapter 1. The Riemann Integral

The Riemann Integral

1.1 Definition: A **partition** of the closed interval $[a, b]$ is a set $X = \{x_0, x_1, \cdots, x_n\}$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\,.$$

The intervals $[x_{i-1}, x_i]$ are called the **subintervals** of $[a, b]$, and we write

$$\Delta_i x = x_i - x_{i-1}$$

for the size of the i^{th} subinterval. Note that

$$\sum_{i=1}^{n} \Delta_i x = b - a\,.$$

The **size** of the partition X, denoted by $|X|$ is

$$|X| = \max\left\{\Delta_i x \middle| 1 \leq i \leq n\right\}.$$

1.2 Definition: Let X be a partition of $[a, b]$, and let $f : [a, b] \to \mathbb{R}$ be bounded. A **Riemann sum** for f on X is a sum of the form

$$S = \sum_{i=1}^{n} f(t_i)\Delta_i x \qquad \text{for some } t_i \in [x_{i-1}, x_i]\,.$$

The points t_i are called **sample points**.

1.3 Definition: Let $f : [a, b] \to \mathbb{R}$ be bounded. We say that f is **(Riemann) integrable** on $[a, b]$ when there exists a number I with the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition X of $[a, b]$ with $|X| < \delta$ we have $|S - I| < \epsilon$ for every Riemann sum for f on X, that is

$$\left|\sum_{i=1}^{n} f(t_i)\Delta_i x - I\right| < \epsilon\,.$$

for every choice of $t_i \in [x_{i-1}, x_i]$ The number I can be shown to be unique. It is called the **(Riemann) integral** of f on $[a, b]$, and we write

$$I = \int_a^b f \text{ , or } I = \int_a^b f(x)\,dx\,.$$

1.4 Example: Show that the constant function $f(x) = c$ is integrable on any interval $[a, b]$ and we have $\displaystyle\int_a^b c\ dx = c(b - a)$.

Solution: The solution is left as an exercise.

1.5 Example: Show that the identity function $f(x) = x$ is integrable on any interval $[a, b]$, and we have $\displaystyle\int_a^b x\ dx = \frac{1}{2}(b^2 - a^2)$.

Solution: Let $\epsilon > 0$. Choose $\delta = \frac{2\epsilon}{b-a}$. Let X be any partition of $[a,b]$ with $|X| < \delta$. Let $t_i \in [x_{i-1}, x_i]$ and set $S = \sum_{i=1}^{n} f(t_i)\Delta_i x = \sum_{i=1}^{n} t_i \Delta_i x$. We must show that $|S - \frac{1}{2}(b^2 - a^2)| < \epsilon$. Notice that

$$\begin{aligned}
\sum_{i=1}^{n}(x_i + x_{i-1})\Delta_i x &= \sum_{i=1}^{n}(x_i + x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n} {x_i}^2 - {x_{i-1}}^2 \\
&= ({x_1}^2 - {x_0}^2) + ({x_2}^2 - {x_1}^2) + \cdots + ({x_{n-1}}^2 - {x_{n-2}}^2) + ({x_n}^2 - {x_{n-1}}^2) \\
&= -{x_0}^2 + ({x_1}^2 - {x_1}^2) + \cdots + ({x_{n-1}}^2 - {x_{n-1}}^2) + {x_n}^2 \\
&= {x_n}^2 - {x_0}^2 = b^2 - a^2
\end{aligned}$$

and that when $t_i \in [x_{i-1}, x_i]$ we have $\left|t_i - \frac{1}{2}(x_i + x_{i-1})\right| \le \frac{1}{2}(x_i - x_{i-1}) = \frac{1}{2}\Delta_i x$, and so

$$\begin{aligned}
\left|S - \tfrac{1}{2}(b^2 - a^2)\right| &= \left|\sum_{i=1}^{n} t_i \Delta_i x - \tfrac{1}{2}\sum_{i=1}^{n}(x_i + x_{i-1})\Delta_i x\right| \\
&= \left|\sum_{i=1}^{n}\left(t_i - \tfrac{1}{2}(x_i + x_{i+1})\right)\Delta_i x\right| \\
&\le \sum_{i=1}^{n}\left|t_i - \tfrac{1}{2}(x_i + x_{i+1})\right|\Delta_i x \\
&\le \sum_{i=1}^{n} \tfrac{1}{2}\Delta_i x \Delta_i x \le \sum_{i=1}^{n} \tfrac{1}{2}\delta \Delta_i x \\
&= \tfrac{1}{2}\delta(b-a) = \epsilon\,.
\end{aligned}$$

1.6 Example: Let $f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q}\,. \end{cases}$ Show that f is not integrable on $[0,1]$.

Solution: Suppose, for a contradiction, that f is integrable on $[0,1]$, and write $I = \int_0^1 f$. Let $\epsilon = \frac{1}{2}$. Choose δ so that for every partition X with $|X| < \delta$ we have $|S - I| < \frac{1}{2}$ for every Riemann sum S for f on X. Choose a partition X with $|X| < \delta$. Let $S_1 = \sum_{i=1}^{n} f(t_i)\Delta_i x$ where each $t_i \in [x_{i-1}, x_i]$ is chosen with $t_i \in \mathbb{Q}$, and let $S_2 = \sum_{i=1}^{n} f(s_i)\Delta_i x$ where each $s_i \in [x_{i-1}, x_i]$ is chosen with $s_i \notin \mathbb{Q}$. Note that we have $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$. Since each $t_i \in \mathbb{Q}$ we have $f(t_i) = 1$ and so $S_1 = \sum_{i=1}^{n} f(t_i)\Delta_i x = \sum_{i=1}^{n} \Delta_i x = 1 - 0 = 1$, and since each $s_i \notin \mathbb{Q}$ we have $f(s_i) = 0$ and so $S_2 = \sum_{i=1}^{n} f(s_i)\Delta_i x = 0$. Since $|S_1 - I| < \frac{1}{2}$ we have $|1 - I| < \frac{1}{2}$ and so $\frac{1}{2} < I < \frac{3}{2}$, and since $|S_2 - I| < \frac{1}{2}$ we have $|0 - I| < \frac{1}{2}$ and so $-\frac{1}{2} < I < \frac{1}{2}$, giving a contradiction.

Integrals of Continuous Functions

1.7 Theorem: *(Continuous Functions are Integrable) Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a,b]$.*

Proof: We omit the proof, which is quite difficult.

1.8 Note: Let f be integrable on $[a,b]$. Let X_n be any sequence of partitions of $[a,b]$ with $\lim\limits_{n\to\infty} |X_n| = 0$. Let S_n be any Riemann sum for f on X_n. Then $\{S_n\}$ converges with

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx\,.$$

Proof: Write $I = \int_a^b f$. Given $\epsilon > 0$, choose $\delta > 0$ so that for every partition X of $[a,b]$ with $|X| < \delta$ we have $|S - I| < \epsilon$ for every Riemann sum S for f on X, and then choose N so that $n > N \Longrightarrow |X_n| < \delta$. Then we have $n > N \Longrightarrow |S_n - I| < \epsilon$.

1.9 Note: Let f be integrable on $[a,b]$. If we let X_n be the partition of $[a,b]$ into n equal-sized subintervals, and we let S_n be the Riemann sum on X_n using right-endpoints, then by the above note we obtain the formula

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x \text{ , where } x_{n,i} = a + \tfrac{b-a}{n}\, i \text{ and } \Delta_{n,i}x = \tfrac{b-a}{n}\,.$$

1.10 Example: Find $\displaystyle\int_0^2 2^x\,dx$.

Solution: Let $f(x) = 2^x$. Note that f is continuous and hence integrable, so we have

$$\begin{aligned}
\int_0^2 2^x\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(\tfrac{2i}{n}\right)\left(\tfrac{2}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^n 2^{2i/n}\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty} \frac{2\cdot 4^{1/n}}{n} \cdot \frac{4-1}{4^{1/n}-1} \text{ , by the formula for the sum of a geometric sequence} \\
&= \Big(\lim_{n\to\infty} 6\cdot 4^{1/n}\Big)\Big(\lim_{n\to\infty} \frac{1}{n\left(4^{1/n}-1\right)}\Big) = 6 \lim_{n\to\infty} \frac{\frac{1}{n}}{4^{1/n}-1} = 6 \lim_{x\to 0} \frac{x}{4^x - 1} \\
&= 6 \lim_{x\to 0} \frac{1}{\ln 4\cdot 4^x} \text{ , by l'Hôpital's Rule} \\
&= \tfrac{6}{\ln 4} = \tfrac{3}{\ln 2}\,.
\end{aligned}$$

1.11 Lemma: *(Summation Formulas) We have*

$$\sum_{i=1}^{n} 1 = n \ , \ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \ , \ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \ , \ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Proof: These formulas could be proven by induction, but we give a more constructive proof. It is obvious that $\sum\limits_{i=1}^{n} 1 = 1+1+\cdots 1 = n$. To find $\sum\limits_{i=1}^{n} i$, consider $\sum\limits_{n=1}^{n} \left(i^2 - (i-1)^2\right)$. On the one hand, we have

$$\begin{aligned}\sum_{i=1}^{n} \left(i^2-(i-1)^2\right) &= (1^2-0^2)+(2^2-1^2)+\cdots+((n-1)^2-(n-2)^2)+(n^2-(n-1)^2) \\ &= -0^2+(1^2-1^2)+(2^2-2^2)+\cdots+((n-1)^2-(n-1)^2)+n^2 \\ &= n^2\end{aligned}$$

and on the other hand,

$$\sum_{i=1}^{n} \left(i^2-(i-1)^2\right) = \sum_{i=1}^{n} \left(i^2-(i^2-2i+1)\right) = \sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

Equating these gives $n^2 = 2\sum\limits_{i=1}^{n} i - \sum\limits_{i=1}^{n} 1$ and so

$$2\sum_{i=1}^{n} i = n^2 + \sum_{i=1}^{n} 1 = n^2+n = n(n+1)\,,$$

as required. Next, to find $\sum\limits_{n=1}^{\infty} i^2$, consider $\sum\limits_{i=1} \left(i^3-(i-1)^3\right)$. On the one hand we have

$$\begin{aligned}\sum_{i=1}^{n} \left(i^3-(i-1)^3\right) &= (1^3-0^3)+(2^3-1^3)+(3^3-2^3)+\cdots+(n^3-(n-1)^3) \\ &= -0^3+(1^3-1^3)+(2^3-2^3)+\cdots+((n-1)^3-(n-1)^3)+n^3 \\ &= n^3\end{aligned}$$

and on the other hand,

$$\begin{aligned}\sum_{i=1}^{n} \left(i^3-(i-1)^3\right) &= \sum_{i=1}^{n} \left(i^3-(i^3-3i^2+3i-1)\right) \\ &= \sum_{i=1}^{n}(3i^2-3i+1) = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\,.\end{aligned}$$

Equating these gives $n^3 = 3\sum\limits_{i=1}^{n} i^2 - 3\sum\limits_{i=1}^{n} i + \sum\limits_{i=1}^{n} 1$ and so

$$6\sum_{i=1}^{n} i^2 = 2n^3 + 6\sum_{i=1}^{n} i - 2\sum_{i=1}^{n} 1 = 2n^3+3n(n+1)-2n = n(n+1)(2n+1)$$

as required. Finally, to find $\sum\limits_{i=1}^{n} i^3$, consider $\sum\limits_{i=1}^{n} \left(i^4-(i-1)^4\right)$. On the one hand we have

$$\sum_{i=1}^{n} \left(i^4-(i-1)^4\right) = n^4\,,$$

(as above) and on the other hand we have

$$\sum_{i=1}^{n} \left(i^4-(i-1)^4\right) = \sum_{i=1}^{n}(4i^3-6i^2+4i-1) = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\,.$$

Equating these gives $n^4 = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$ and so

$$\begin{aligned} 4\sum_{i=1}^{n} i^3 &= n^4 + 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= n^4 + n(n+1)(2n+1) - 2n(n+1) + n \\ &= n^4 + 2n^3 + n^2 = n^2(n+1)^2\,, \end{aligned}$$

as required.

1.12 Example: Find $\displaystyle\int_1^3 x + 2x^3\; dx$.

Solution: Let $f(x) = x + 2x^3$. Then

$$\begin{aligned} \int_1^3 x + 2x^3\; dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i})\Delta_{n,i}x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \tfrac{2}{n}\,i\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \tfrac{2}{n}\,i\right) + 2\left(1 + \tfrac{2}{n}\,i\right)^3\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \tfrac{2}{n}\,i + 2\left(1 + \tfrac{6}{n}\,i + \tfrac{12}{n^2}\,i^2 + \tfrac{8}{n^3}\,i^3\right)\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{6}{n} + \tfrac{28}{n^2}\,i + \tfrac{48}{n^3}\,i^2 + \tfrac{32}{n^4}\,i^3\right) \\ &= \lim_{n\to\infty} \left(\tfrac{6}{n}\sum_{i=1}^{n} 1 + \tfrac{28}{n^2}\sum_{i=1}^{n} i + \tfrac{48}{n^3}\sum_{i=1}^{n} i^2 + \tfrac{32}{n^4}\sum_{i=1}^{n} i^3\right) \\ &= \lim_{n\to\infty} \left(\tfrac{6}{n}\cdot n + \tfrac{28}{n^2}\cdot\tfrac{n(n+1)}{2} + \tfrac{48}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} + \tfrac{32}{n^4}\cdot\tfrac{n^2(n+1)^2}{4}\right) \\ &= 6 + \tfrac{28}{2} + \tfrac{48\cdot 2}{6} + \tfrac{32}{4} = 44\,. \end{aligned}$$

Basic Properties of Integrals

1.13 Theorem: *(Linearity) Let f and g be integrable on $[a,b]$ and let $c \in \mathbb{R}$. Then $f+g$ and cf are both integrable on $[a,b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c\int_a^b f\,.$$

Proof: The proof is left as an exercise.

1.14 Theorem: *(Comparison) Let f and g be integrable on $[a,b]$. If $f(x) \le g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f \le \int_a^b g\,.$$

Proof: The proof is left as an exercise.

1.15 Theorem: *(Additivity) Let $a<b<c$ and let $f:[a,c]\to\mathbb{R}$ be bounded. Then f is integrable on $[a,c]$ if and only if f is integrable both on $[a,b]$ and on $[b,c]$, and in this case*

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

Proof: We omit the proof, which is quite difficult.

1.16 Definition: We define $\displaystyle\int_a^a f = 0$ and for $a<b$ we define $\displaystyle\int_b^a f = -\int_a^b f$.

1.17 Note: Using the above definition, the Additivity Theorem extends to the case that $a,b,c \in \mathbb{R}$ are not in increasing order: for any $a,b,c\in\mathbb{R}$, if f is integrable on $\big[\min\{a,b,c\},\max\{a,b,c\}\big]$ then

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

1.18 Theorem: *(Absolute Value) Let f be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \le \int_a^b |f|\,.$$

Proof: We omit the proof, which is quite difficult.

1.19 Notation: For a function F, defined on an interval containing $[a,b]$, we write

$$\Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

1.20 Theorem: *(The Fundamental Theorem of Calculus)*
(1) Let f be integrable on $[a,b]$. Define $F:[a,b]\to\mathbb{R}$ by

$$F(x) = \int_a^x f = \int_a^x f(t)\,dt\,.$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x\in[a,b]$ then F is differentiable at x and

$$F'(x) = f(x)\,.$$

(2) Let f be integrable on $[a,b]$. Let F be differentiable on $[a,b]$ with $F' = f$. Then

$$\int_a^b f = \Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

Proof: (1) Let M be an upper bound for $|f|$ on $[a,b]$. For $a \le x,y \le b$ we have

$$\big|F(y)-F(x)\big| = \left|\int_a^y f - \int_a^x f\right| = \left|\int_x^y f\right| \le \left|\int_x^y |f|\right| \le \left|\int_x^y M\right| = M|y-x|$$

so given $\epsilon > 0$ we can choose $\delta = \frac{\epsilon}{M}$ to get

$$|y-x|<\delta \Longrightarrow \big|F(y)-F(x)\big| \le M|y-x| < M\delta = \epsilon\,.$$

Thus F is continuous on $[a,b]$. Now suppose that f is continuous at the point $x\in[a,b]$. Note that for $a\le x,y\le b$ with $x\ne y$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &= \left|\frac{\int_a^y f - \int_a^x f}{y-x} - f(x)\right| \\
&= \left|\frac{\int_x^y f}{y-x} - \frac{\int_x^y f(x)}{y-x}\right| \\
&= \frac{1}{|y-x|}\left|\int_x^y \big(f(t)-f(x)\big)\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right|\,.
\end{aligned}$$

Given $\epsilon>0$, since f is continuous at x we can choose $\delta>0$ so that

$$|y-x|<\delta \Longrightarrow \big|f(y)-f(x)\big| < \epsilon$$

and then for $0<|y-x|<\delta$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \epsilon\,dt\right| = \frac{1}{|y-x|}\,\epsilon|y-x| = \epsilon\,.
\end{aligned}$$

and thus we have $F'(x) = f(x)$ as required.

(2) Let f be integrable on $[a,b]$. Suppose that F is differentiable on $[a,b]$ with $F'=f$. Let $\epsilon>0$ be arbitrary. Choose $\delta>0$ so that for every partition X of $[a,b]$ with $|X|<\delta$ we have $\left|\int_a^b f-\sum_{i=1}^n f(t_i)\Delta_i x\right|<\epsilon$ for every choice of sample points $t_i\in[x_{i-1},x_i]$. Choose sample points $t_i\in[x_{i-1},x_i]$ as in the Mean Value Theorem so that

$$F'(t_i)=\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}\,,$$

that is $f(t_i)\Delta_i x=F(x_i)-F(x_{i-1})$. Then $\left|\int_a^b f-\sum_{i=1}^n f(t_i)\Delta_i x\right|<\epsilon$, and

$$\begin{aligned}\sum_{i=1}^n f(t_i)\Delta_i x &= \sum_{i=1}^n \big(F(x_i)-F(x_{i-1}\big) \\ &= \big(F(x_1)-F(x)\big)+\big(F(x_2)-F(x_1)\big)+\cdots+\big(F(n-1)-F(x_n)\big) \\ &= -F(x)+\big(F(x_1)-F(x_1)\big)+\cdots+\big(F(x_{n-1})-F(x_{n-1})\big)+F(x_n) \\ &= F(x_n)-F(x)=F(b)-F(a)\,.\end{aligned}$$

and so $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|<\epsilon$. Since ϵ was arbitrary, $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|=0$.

1.21 Definition: A function F such that $F'=f$ on an interval is called an **antiderivative** of f on the interval.

1.22 Note: If $G'=F'=f$ on an interval, then $(G-F)'=0$, and so $G-F$ is constant on the interval, that is $G=F+c$ for some constant c.

1.23 Notation: We write

$$\int f=F+c\ ,\text{ or }\int f(x)\,dx=F(x)+c$$

when F is an antiderivative of f on an interval, so that the antiderivatives of f on the interval are the functions of the form $G=F+c$ for some constant c.

1.24 Example: Find $\displaystyle\int_0^{\sqrt{3}}\frac{dx}{1+x^2}$.

Solution: We have $\displaystyle\int\frac{dx}{1+x^2}=\tan^{-1}x+c$, since $\displaystyle\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$, and so by Part 2 of the Fundamental Theorem of Calculus, we have

$$\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_0^{\sqrt{3}}=\tan^{-1}\sqrt{3}-\tan^{-1}0=\tfrac{\pi}{3}\,.$$