Chapter 2. Methods of Integration

Basic Integrals

2.1 Note: We have the following list of **Basic Integrals**

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c\ ,\ \text{for } p \neq -1 \qquad \int \sec^2 x\ dx = \tan x + c$$

$$\int \frac{dx}{x} = \ln|x| + c \qquad \int \sec x \tan x\ dx = \sec x + c$$

$$\int e^x\,dx = e^x + c \qquad \int \tan x\ dx = \ln|\sec x| + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c \qquad \int \sec x\ dx = \ln|\sec x + \tan x| + c$$

$$\int \ln x\ dx = x\ln x - x + c \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

$$\int \sin x\ dx = -\cos x + c \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$$

$$\int \cos x\ dx = \sin x + c \qquad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$$

Proof: Each of these equalities is easy to verify by taking the derivative of the right side. For example, we have $\int \ln x\ dx = x\ln x - x + c$ since $\frac{d}{dx}(x\ln x - x) = 1\cdot\ln x + x\cdot\frac{1}{x} - 1 = \ln x$, and we have $\int \tan x\ dx = \ln|\sec x| + c$ since $\frac{d}{dx}(\ln|\sec x|) = \frac{\sec x\tan x}{\sec x} = \tan x$, and we have $\int \sec x\ dx = \ln|\sec x + \tan x| + c$ since $\frac{d}{dx}(\ln|\sec x + \tan x|) = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x$.

2.2 Example: Find $\displaystyle\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx$.

Solution: By the Fundamental Theorem of Calculus and Linearity, we have

$$\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx = \int_1^4 x^{3/2} - 5x^{-1/2}\,dx = \left[\tfrac{2}{5}x^{5/2} - 10x^{1/2}\right]_1^4 = \left(\tfrac{64}{5} - 20\right) - \left(\tfrac{2}{5} - 10\right) = \tfrac{12}{5}\,.$$

Substitution

2.3 Theorem: *(Substitution, or Change of Variables) Let* $u = g(x)$ *be differentiable on an interval and let* $f(u)$ *be continuous on the range of* $g(x)$*. Then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

and

$$\int_{x=a}^{b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{g(b)} f(u)\,du\,.$$

Proof: Let $F(u)$ be an antiderivative of $f(u)$ so $F'(u) = f(u)$ and $\int f(u)\,du = F(u) + c$. Then from the Chain Rule, we have $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$, and so

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c = F(u) + c = \int f(u)\,du$$

and

$$\begin{aligned}\int_{x=a}^{b} f(g(x))g'(x)\,dx &= \Big[F(g(x))\Big]_{x=a}^{b} = F(g(b)) - F(g(a)) \\ &= \Big[F(u)\Big]_{u=g(a)}^{g(b)} = \int_{u=g(a)}^{g(b)} f(u)\,du\,.\end{aligned}$$

2.4 Notation: For $u = g(x)$ we write $du = g'(x)\,dx$. More generally, for $f(u) = g(x)$ we write $f'(u)\,du = g'(x)\,dx$. This notation makes the above theorem easy to remember and to apply.

2.5 Example: Find $\int \sqrt{2x+3}\,dx$.

Solution: Make the substitution $u = 2x + 3$ so $du = 2dx$. Then

$$\int \sqrt{2x+3}\,dx = \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{3}\,u^{3/2} + c = \tfrac{1}{3}(2x+3)^{3/2} + c\,.$$

(In applying the Substitution Rule, we used $u = g(x) = 2x + 3$ and $f(u) = \sqrt{u} = u^{1/2}$, but the notation $du = g'(x)\,dx$ allows us to avoid explicit mention of the function $f(u)$ in our solution).

2.6 Example: Find $\int x\,e^{x^2}\,dx$.

Solution: Make the substitution $u = x^2$ so $du = 2x\,dx$. Then

$$\int x\,e^{x^2}\,dx = \int \tfrac{1}{2}e^u\,du = \tfrac{1}{2}e^u + c = \tfrac{1}{2}e^{x^2} + c\,.$$

2.7 Example: Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Solution: Let $u = \ln x$ so $du = \dfrac{1}{x}\,dx$. Then

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \tfrac{1}{2}u^2 + c = \tfrac{1}{2}(\ln x)^2 + c\,.$$

2.8 Example: Find $\displaystyle\int \tan x\ dx$.

Solution: We have $\tan x = \dfrac{\sin x}{\cos x}$. Let $u = \cos x$ so $du = -\sin x\ dx$. Then

$$\int \tan x\ dx = \int \frac{\sin x\ dx}{\cos x} = \int \frac{-du}{u} = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c\,.$$

2.9 Example: Find $\displaystyle\int \frac{dx}{x+\sqrt{x}}$.

Solution: Let $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = \int \frac{2\,du}{u+1}\,.$$

Now let $v = u + 1$ do $dv = du$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\,du}{u+1} = \int \frac{2}{v}\,dv = 2\ln|v| + c = 2\ln|u+1| + c = 2\ln(\sqrt{x}+1) + c\,.$$

2.10 Example: Find $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{2x^2+1}}$.

Solution: Let $u = 2x^2 + 1$ so $du = 4x\,dx$. Then

$$\int_{x=0}^2 \frac{x\,dx}{\sqrt{2x^2+1}} = \int_{u=1}^9 \frac{\frac{1}{4}\,du}{\sqrt{u}} = \int_1^9 \tfrac{1}{4}\,u^{-1/2}\,du = \Big[\tfrac{1}{2}\,u^{1/2}\Big]_1^9 = \tfrac{3}{2} - \tfrac{1}{2} = 1\,.$$

2.11 Example: Find $\displaystyle\int_0^1 \frac{dx}{1+3x^2}$.

Solution: Let $u = \sqrt{3}\,x$ so $du = \sqrt{3}\,dx$. Then

$$\int_0^1 \frac{dx}{1+3x^2} = \int_0^{\sqrt{3}} \frac{\frac{1}{\sqrt{3}}\,du}{1+u^2} = \Big[\tfrac{1}{\sqrt{3}}\tan^{-1} u\Big]_0^{\sqrt{3}} = \tfrac{1}{\sqrt{3}}\,\tfrac{\pi}{3} = \tfrac{\pi}{3\sqrt{3}}\,.$$

Integration by Parts

2.12 Theorem: *(Integration by Parts) Let $f(x)$ and $g(x)$ be differentiable in an interval. Then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

so

$$\int_{x=a}^{b} f(x)g'(x)\,dx = \left[f(x)g(x) - \int g(x)f'(x)\,dx\right]_{x=a}^{b}.$$

Proof: By the Product Rule, we have $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ and so

$$\int f'(x)g(x) + f(x)g'(x)\ dx = f(x)g(x) + c\,,$$

which can be rewritten as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx\,.$$

(We do not need to include the arbitrary constant c since there is now an integral on both sides of the equation).

2.13 Notation: If we write $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$ and $dv = g'(x)\,dx$, then the top formula in the above theorem becomes

$$\int u\,dv = uv - \int v\,du\,.$$

2.14 Note: To find the integral of a polynomial multiplied by an exponential function or a trigonometric function, try Integrating by parts with u equal to the polynomial (you may need to integrate by parts repeatedly if the polynomial is of high degree).

To integrate a polynomial (or an algebraic) function times a logarithmic or inverse trigonometric function, try integrating by parts letting u be the logarithmic or inverse trigonometric function.

To integrate an exponential function times a sine or cosine function, try integrating by parts twice, letting u be the exponential function both times.

2.15 Example: Find $\int x\sin x\ dx$.

Solution: Integrate by parts using $u = x$, $du = dx$, $v = -\cos x$ and $dv = \sin x\ dx$ to get

$$\int x\sin x\ dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\ dx = -x\cos x + \sin x + c\,.$$

2.16 Example: Find $\displaystyle\int (x^2+1)e^{2x}\,dx$.

Solution: Integrate by parts using $u = x^2+1$, $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$ and $dv = e^{2x}\,dx$ to get

$$\int (x^2+1)e^{2x}\,dx = \int u\,dv = uv - \int v\,du = \tfrac{1}{2}(x^2+1)e^{2x} - \int x\,e^{2x}\,dx\,.$$

To find $\displaystyle\int x\,e^{2x}\,dx$ we integrate by parts again, this time using $u_2 = x$, $du_2 = dx$, $v_2 = \frac{1}{2}e^{2x}$ and $dv_2 = e^{2x}\,dx$ to get

$$\begin{aligned}\int (x^2+1)e^{2x}\,dx &= \tfrac{1}{2}(x^2+1)e^{2x} - \int x\,e^{2x}\,dx \\ &= \tfrac{1}{2}(x^2+1)e^{2x} - \left(\tfrac{1}{2}\,xe^{2x} - \int \tfrac{1}{2}e^{2x}\,dx\right) \\ &= \tfrac{1}{2}(x^2+1)e^{2x} - \left(\tfrac{1}{2}\,xe^{2x} - \tfrac{1}{4}e^{2x}\right) + c \\ &= \tfrac{1}{4}\left(2x^2-2x+3\right)e^{2x} + c\end{aligned}$$

2.17 Example: Find $\displaystyle\int \ln x\;dx$.

Solution: Integrate by parts using $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $v = x$ and $dv = dx$ to get

$$\int \ln x\;dx = x\ln x - \int 1\,dx = x\ln x - x + c\,.$$

2.18 Example: Find $\displaystyle\int_1^4 \sqrt{x}\,\ln x\;dx$.

Solution: Integrate by parts using $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$ and $dv = x^{1/2}\,dx$ to get

$$\begin{aligned}\int_1^4 \sqrt{x}\,\ln x\;dx &= \left[\tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx\right]_1^4 = \left[\tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2}\right]_1^4 \\ &= \left(\tfrac{16}{3}\ln 4 - \tfrac{32}{9}\right) - \left(\tfrac{2}{3}\ln 1 - \tfrac{4}{9}\right) = \tfrac{16}{3}\ln 4 - \tfrac{28}{9}\,.\end{aligned}$$

2.19 Example: Find $\displaystyle\int e^x \sin x\;dx$

Solution: Write $I = \displaystyle\int e^x\sin x\,dx$. Integrate by parts twice, first using $u_1 = e^x$, $du = e^x\,dx$, $v = -\cos x$ and $dv = \sin x\;dx$, and next using $u_2 = e^x$, $du_2 = e^x\,dx$, $v_2 = \sin x$ and $dv_2 = \cos x\;dx$ to get

$$\begin{aligned}I &= -e^x\cos x + \int e^x\cos x\;dx \\ &= -e^x\cos x + \left(e^x\sin x - \int e^x\sin x\;dx\right)\,. \\ &= -e^x\cos x + e^x\sin x - I\end{aligned}$$

Thus $2I = -e^x\cos x + e^x\sin x + c$ and so $I = \frac{1}{2}(\sin x - \cos x)e^x + d$.

2.20 Example: Let $n \geq 2$ be an integer. Find a formula for $\int \sin^n x\ dx$ in terms of $\int \sin^{n-2} x\ dx$, and hence find $\int \sin^2 x\ dx$ and $\int \sin^4 x\ dx$.

Solution: Let $I = \int \sin^n x\ dx = \int \sin^{n-1} x \sin x\ dx$. Integrate by parts using $u = \sin^{n-1} x$, $du = (n-1)\sin^{n-2} x \cos x\, dx$, $v = -\cos x$ and $dv = \sin x\, dx$ to get

$$\begin{aligned} I &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\ dx \\ &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x (1-\sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)I\,. \end{aligned}$$

Add $(n-1)I$ to both sides to get $nI = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\ dx$, that is

$$\int \sin^n x\ dx = -\tfrac{1}{n}\sin^{n-1} x \cos x + \tfrac{n-1}{n}\int \sin^{n-2} x\ dx\,.$$

In particular, when $n = 2$ we get

$$\int \sin^2 x\ dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int 1\, dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + c$$

and when $n = 4$ we get

$$\int \sin^4 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + c\,.$$

2.21 Example: Let $n \geq 2$ be an integer. Find a formula for $\int \sec^n x\ dx$ in terms of $\int \sec^{n-2} x\ dx$, and hence find $\int \sec^3 x\ dx$.

Solution: Let $I = \int \sec^n x\ dx = \int \sec^{n-2} x \sec^2 x\ dx$. Using Integrate by Parts with $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x \tan x\, dx$, $v = \tan x$ and $dv = \tan x\, dx$, we obtain

$$\begin{aligned} I &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\ dx \\ &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x (\sec^2 x - 1)\ dx \\ &= \sec^{n-2} x \tan x - (n-2)I + (n-2)\int \sec^{n-2} x\ dx \end{aligned}$$

Add $(n-2)I$ to both sides to get $(n-1)I = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\ dx$, that is

$$\int \sec^n x\ dx = \tfrac{1}{n-1}\sec^{n-2} x \tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x\ dx\,.$$

In particular, when $n = 3$ we get

$$\int \sec^3 x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln\big|\sec x + \tan x\big| + c$$

2.22 Note: To find $\int f(\sin x)\cos^{2n+1}x\ dx$, write $\cos^{2n+1}x=(1-\sin^2 x)^n\cos x$ then try the substitution $u=\sin x$, $du=\cos x\,dx$.

To find $\int f(\cos x)\sin^{2n+1}x\ dx$, write $\sin^{2n+1}x=(1-\cos^2 x)^n\sin x$ then try the substitution $u=\cos x$, $du=-\sin x\ dx$.

To find $\int \sin^{2m}x\cos^{2n}x\ dx$, try using the trigonometric identities $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ and $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$. Alternatively, write $\cos^{2n}x=(1-\sin^2 x)^n$ and use the formula from Example 2.20.

To find $\int f(\tan x)\sec^{2n+2}x\ dx$, write $\sec^{2n+2}x=(1+\tan^2 x)^n\sec^2 x\,dx$ and try the substitution $u=\tan x$, $du=\sec^2 x\,dx$.

To find $\int f(\sec x)\tan^{2n+1}x\ dx$, write $\tan^{2n+1}x=\dfrac{(\sec^2 x-1)^n}{\sec x}\sec x\tan x\ dx$ and try the substitution $u=\sec x$, $du=\sec x\tan x\ dx$.

To find $\int \sec^{2n+1}x\tan^{2n}x\ dx$, write $\tan^{2n}x=(\sec^2 x-1)^n$ and use the formula from Example 2.21.

2.23 Example: Find $\displaystyle\int_0^{\pi/3}\frac{\sin^3 x}{\cos^2 x}\,dx$.

Solution: Make the substitution $u=\cos x$ so $du=-\sin x\,dx$. Then

$$\int_0^{\pi/3}\frac{\sin^3 x}{\cos^2 x}\,dx=\int_0^{\pi/3}\frac{(1-\cos^2 x)\sin x\,dx}{\cos^2 x}=\int_1^{1/2}-\frac{(1-u^2)\,du}{u^2}=\int_1^{1/2}-\frac{1}{u^2}+1\ du$$
$$=\Big[\tfrac{1}{u}+u\Big]_1^{1/2}=\big(2+\tfrac{1}{2}\big)-(1+1)=\tfrac{1}{2}\,.$$

2.24 Example: Find $\int \sin^6 x\ dx$.

Solution: We could use the method of example 2.20, but we choose instead to use the half-angle formulas. We have

$$\int_0^{\pi/4}\sin^6 x\ dx=\int_0^{\pi/4}\big(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\big)^3\,dx=\int_0^{\pi/4}\tfrac{1}{8}-\tfrac{3}{8}\cos 2x+\tfrac{3}{8}\cos^2 2x-\tfrac{1}{8}\cos^3 2x\ dx$$
$$=\int_0^{\pi/4}\tfrac{1}{8}-\tfrac{3}{8}\cos 2x+\tfrac{3}{8}\big(\tfrac{1}{2}+\tfrac{1}{2}\cos 4x\big)-\tfrac{1}{8}\big(1-\sin^2 2x\big)\cos 2x\ dx$$
$$=\int_0^{\pi/4}\tfrac{5}{16}-\tfrac{1}{2}\cos 2x+\tfrac{3}{16}\cos 4x+\tfrac{1}{8}\sin^2 2x\cos 2x\ dx$$
$$=\Big[\tfrac{5}{15}x-\tfrac{1}{4}\sin 2x+\tfrac{3}{64}\sin 4x+\tfrac{1}{48}\sin^3 2x\Big]_0^{\pi/4}$$
$$=\tfrac{5\pi}{64}-\tfrac{1}{4}+\tfrac{1}{48}=\tfrac{5\pi}{64}-\tfrac{11}{48}\,.$$

2.25 Example: Find $\displaystyle\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x+1}}\,dx$.

Solution: Make the substitution $u=\tan x$ so $du=\sec^2 x\,dx$. Then

$$\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x+1}}\,dx = \int_0^{\pi/4} \frac{(\tan^2 x+1)\sec^2 x\,dx}{\sqrt{\tan x+1}} = \int_0^1 \frac{(u^2+1)\,du}{\sqrt{u+1}}$$

Now make the substitution $v=u+1$ so $u=v-1$ and $du=dv$. Then

$$\begin{aligned}\int_0^1 \frac{u^2+1}{\sqrt{u+1}}\,du &= \int_1^2 \frac{(v-1)^2+1}{\sqrt{v}}\,dv = \int_1^2 v^{3/2}-2\,v^{1/2}+2\,v^{-1/2}\;dv\\ &= \Big[\tfrac{2}{5}\,v^{5/2}-\tfrac{4}{3}\,v^{3/2}+4\,v^{1/2}\Big]_1^2 = \big(\tfrac{2\cdot4\sqrt2}{5}-\tfrac{4\cdot2\sqrt2}{3}+4\sqrt2\big)-\big(\tfrac25-\tfrac43+4\big)\\ &= \tfrac{(24-40+60)\sqrt2}{15)}-\tfrac{6-20+60}{15}=\tfrac{44\sqrt2-46}{15}\,.\end{aligned}$$

2.26 Example: Find $\displaystyle\int_0^{\pi/4} \tan^4 x\;dx$.

Solution: Note first that

$$\tan^4 x=\tan^2 x(\sec^2 x-1)=\tan^2 x\sec^2 x-\tan^2 x=\tan^2 x\sec^2 x-\sec^2 x+1\,.$$

To find $\displaystyle\int \tan^2 x\sec^2 x\;dx$, make the substitution $u=\tan\theta$, $du=\sec^2\theta\,d\theta$ to get

$$\int \tan^2 x\sec^2 x\;dx=\int u^2\,du=\tfrac13 u^3+c=\tfrac13\tan^3 x+c\,.$$

Thus we have

$$\begin{aligned}\int_0^{\pi/4}\tan^4 x\,dx &= \int_0^{\pi/4}\tan^2 x\sec^2 x-\sec^2 x+1\\ &= \Big[\tfrac13\,\tan^3 x-\tan x+x\Big]_0^{\pi/4}=\tfrac13-1+\tfrac{\pi}{4}=\tfrac{\pi}{4}-\tfrac23\,.\end{aligned}$$

2.27 Note: To find $\displaystyle\int \sin(ax)\sin(bx)\,dx$, $\displaystyle\int \cos(ax)\cos(bx)\,dx$, or $\displaystyle\int \sin(ax)\cos(bx)\,dx$, use one of the identities

$$\begin{aligned}\cos(A-B)-\cos(A+B)&=2\sin A\sin B\\ \cos(A-B)+\cos(A+B)&=2\cos A\cos B\\ \sin(A-B)+\sin(A+B)&=2\sin A\cos B\,.\end{aligned}$$

2.28 Example: Find $\displaystyle\int_0^{\pi/6}\cos 3x\cos 2x\,dx$.

Solution: Since $2\cos 3x\cos 2x=\cos(3x-2x)+\cos(3x+2x)=\cos x+\cos 5x$, we have

$$\int_0^{\pi/6}\cos 2x\cos 3x\,dx=\int_0^{\pi/6}\tfrac12(\cos x+\cos 5x)\;dx=\Big[\tfrac12\sin x+\tfrac{1}{10}\sin 5x\Big]_0^{\pi/6}=\tfrac14+\tfrac{1}{20}=\tfrac{3}{10}\,.$$

2.29 Note: The **Weirstrass substitution** $u = \tan\frac{x}{2}$, $x = 2\tan^{-1}u$, $dx = \dfrac{2du}{1+u^2}$ converts $\sin x$ and $\cos x$ into rational functions of u: indeed we have $\sin\frac{x}{2} = \frac{u}{\sqrt{1-u^2}}$ and $\cos\frac{x}{2} = \frac{1}{\sqrt{1-u^2}}$ so that $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2u}{1+u^2}$ and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-u^2}{1+u^2}$.

2.30 Example: Find $\displaystyle\int \frac{dx}{1-\cos x}$.

Solution: We use the Weirstrass substitution $u = \tan\frac{x}{2}$, $dx = \frac{2}{1+u^2}\,du$, and $\cos x = \frac{1-u^2}{1+u^2}$ to get

$$\int \frac{dx}{1-\cos x} = \int \frac{\frac{2}{1+u^2}\,du}{1-\frac{1-u^2}{1+u^2}} = \int \frac{2\,du}{(1+u^2)-(1-u^2)} = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\cot\frac{x}{2} + c\,.$$

Inverse Trigonometric Substitution

2.31 Note: To solve an integral involving $\sqrt{a^2+b^2(x+c)^2}$ or $1/(a^2+b^2(x+c)^2)$, try the substitution $\theta=\tan^{-1}\frac{b(x+c)}{a}$ so that $a\tan\theta=b(x+c)$, $a\sec\theta=\sqrt{a^2+b^2(x+c)^2}$ and $a\sec^2\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{a^2-b^2(x+c)^2}$, try the substitution $\theta=\sin^{-1}\frac{b(x+c)}{a}$ so that $a\sin\theta=b(x+c)$, $a\cos\theta=\sqrt{a^2-b^2(x+c)^2}$ and $a\cos\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{b^2(x+c)^2-a^2}$, try the substitution $\theta=\sec^{-1}\frac{b(x+c)}{a}$ so that $a\sec\theta=b(x+c)$, $a\tan\theta=\sqrt{b^2(x+c)^2-a^2}$ and $a\sec\theta\tan\theta\,d\theta=b\,dx$.

2.32 Example: Find $\displaystyle\int_0^1\frac{dx}{(4-3x^2)^{3/2}}$.

Solution: Let $2\sin\theta=\sqrt{3}\,x$ so $2\cos\theta=\sqrt{4-3x^2}$ and $2\cos\theta\,d\theta=\sqrt{3}\,dx$. Then

$$\int_0^1\frac{dx}{(4-3x^2)^{3/2}}=\int_0^{\pi/3}\frac{\frac{2}{\sqrt{3}}\cos\theta\,d\theta}{(2\cos\theta)^3}=\int_0^{\pi/3}\tfrac{1}{4\sqrt{3}}\sec^2\theta\,d\theta=\Big[\tfrac{1}{4\sqrt{3}}\,\tan\theta\Big]_0^{\pi/3}=\tfrac14\,.$$

2.33 Example: Find $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}$.

Solution: Let $\sqrt{3}\,\tan\theta=x$ so $\sqrt{3}\,\sec\theta=\sqrt{x^2+3}$ and $\sqrt{3}\,\sec^2\theta\,d\theta=dx$, and also let $u=\sin\theta$ so $du=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}&=\int_{\pi/6}^{\pi/4}\frac{\sqrt{3}\,\sec^2\theta\,d\theta}{3\tan^2\theta\,\sqrt{3}\sec\theta}=\int_{\pi/6}^{\pi/4}\frac13\,\frac{\sec\theta}{\tan^2\theta}\,d\theta=\int_{\pi/6}^{\pi/4}\frac13\,\frac{\cos\theta\,d\theta}{\sin^2\theta}\\&=\int_{1/2}^{1/\sqrt{2}}\frac{1}{3\,u^2}\,du=\Big[-\frac{1}{3u}\Big]_{1/2}^{1/\sqrt{2}}=-\tfrac{\sqrt{2}}{3}+\tfrac23=\tfrac{2-\sqrt{2}}{3}\,.\end{aligned}$$

2.34 Example: Find $\displaystyle\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx$.

Solution: Let $2\sec\theta=x$ so $2\tan\theta=\sqrt{x^2-4}$ and $2\sec\theta\tan\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx&=\int_0^{\pi/3}\frac{\tan^2\theta\sec\theta\,d\theta}{\sec^2\theta}=\int_0^{\pi/3}\frac{\tan^2\theta}{\sec\theta}\,d\theta=\int_0^{\pi/3}\frac{\sec^2\theta-1}{\sec\theta}\,d\theta\\&=\int_0^{\pi/3}\sec\theta-\cos\theta\;d\theta=\Big[\ln|\sec\theta+\tan\theta|-\sin\theta\Big]_0^{\pi/3}=\ln(2+\sqrt{3})-\tfrac{\sqrt{3}}{2}\,.\end{aligned}$$

2.35 Example: Find $\displaystyle\int_2^3(4x-x^2)^{3/2}\,dx$.

Solution: Let $2\sin\theta=x-2$ so $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^3(4x-x^2)^{3/2}\,dx&=\int_0^{\pi/6}16\,\cos^4\theta\,d\theta=\int_0^{\pi/6}4\,(1+\cos2\theta)^2\,d\theta\\&=\int 4+8\cos2\theta+4\cos^2 2\theta\;d\theta=\int 4+8\cos2\theta+2+2\cos4\theta\;d\theta\\&=\Big[6\theta+4\sin2\theta+\tfrac12\sin4\theta\Big]_0^{\pi/6}=\pi+2\sqrt{3}+\tfrac{\sqrt{3}}{4}=\pi+\tfrac{9\sqrt{3}}{4}\,.\end{aligned}$$

Partial Fractions

2.36 Note: We can find the integral of a rational function $\dfrac{f(x)}{g(x)}$ as follows:

Step 1: use long division to find polynomials $q(x)$ and $r(x)$ with $\deg r(x) < \deg g(x)$ such that $f(x) = g(x)q(x) + r(x)$ for all x, and note that $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$ so

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x) + \frac{r(x)}{g(x)}\,dx\,.$$

(If $\deg f(x) < \deg g(x)$ then $q(x) = 0$ and $r(x) = f(x)$).

Step 2: factor $g(x)$ into linear and irreducible quadratic factors.

Step 3: write $\dfrac{r(x)}{g(x)}$ as a sum of terms so that for each linear factor $(ax+b)^k$ we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

and for each irreducible quadratic factor $(ax^2+bx+c)^k$ we have the k terms

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}\,.$$

Writing $\dfrac{r(x)}{g(x)}$ in this form is called splitting $\dfrac{r(x)}{g(x)}$ into its **partial fractions** decomposition.

Step 4: solve the integral.

2.37 Example: If $g(x) = x(x-1)^3(x^2+2x+3)^2$ then in step 3 we would write

$$\frac{r(x)}{g(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+2x+3} + \frac{Gx+H}{(x^2+2x+3)^2}\,.$$

and then solve for the various constants.

2.38 Example: Find $\displaystyle\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

Solution: In order to get $\dfrac{x-7}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ we need

$$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x-7\,.$$

Equating coefficients gives $A+C=0$, $A+B-2C=1$ and $-2A+2B+C=-7$. Solving these three equations gives $A=1$, $B=-2$ and $C=-1$, and so we have

$$\begin{aligned}\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx &= \int_2^3 \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}\\ &= \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx = \Big[\ln(x-1) + \tfrac{2}{x-1} - \ln(x+2)\Big]_2^3\\ &= (\ln 2 + 1 - \ln 5) - (2 - \ln 4) = \ln\frac{8}{5} - 1\,.\end{aligned}$$

2.39 Example: Find $\displaystyle\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$.

Solution: Use long division of polynomials to show that $\dfrac{x^4-x^3+1}{x^3+x} = x-1+\dfrac{-x^2+x+1}{x^3+x}$. Next, note that to get $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1} = \dfrac{-x^2+x+1}{x^3+x}$ we need $A(x^2+1)+(Bx+C)(x) = -x^2+x+1$. Equating coefficients gives $A+B=-1$, $C=1$ and $A=1$. Solving these three equations gives $A=1$, $B=-2$ and $C=1$. Thus

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx &= \int_1^{\sqrt{3}} x-1+\frac{1}{x}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}\,dx \\
&= \Big[\tfrac{1}{2}x^2 - x + \ln x - \ln(x^2+1) + \tan^{-1}x\Big]_1^{\sqrt{3}} \\
&= \left(\tfrac{3}{2}-\sqrt{3}+\ln\sqrt{3}-\ln 4+\tfrac{\pi}{3}\right)-\left(\tfrac{1}{2}-1-\ln 2+\tfrac{\pi}{4}\right) \\
&= 2-\sqrt{3}+\ln\tfrac{\sqrt{3}}{2}+\tfrac{\pi}{12}\,.
\end{aligned}$$

2.40 Example: Find $I = \displaystyle\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$.

Solution: To get

$$\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-2x+5}+\frac{Ex+F}{(x^2-2x+5)^2} = \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}$$

we need $Ax(x^2-2x+5)^2+B(x^2-2x+5)^2+(Cx+D)(x^2)(x^2-2x+5)+(Ex+F)(x^2) = x^5+x^4-2x^3-2x^2-5x-25$. Expanding the left hand side then equating coefficients gives the 5 equations

$$\begin{aligned}
&A+C=1\ ,\ -4A+B-2C+D=1\ ,\ 14A-4B+5C-2D+E=-2 \\
&-20A+14B+5D+F=-2\ ,\ 25A-20B=-5\ ,\ 25B=-25
\end{aligned}$$

Solving these equations gives $A=-1$, $B=-1$, $C=2$, $D=2$, $E=2$ and $F=-18$, so

$$\begin{aligned}
I &= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x+2}{x^2-2x+5}+\frac{2x-18}{(x^2-2x+5)^2}\,dx \\
&= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x-2+4}{x^2-2x+5}+\frac{2x-2-16}{(x^2-2x+5)^2}\,dx \\
&= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x-2}{x^2-2x+5}+\frac{4}{x^2-2x+5}+\frac{2x-2}{(x^2-2x+5)^2}-\frac{16}{(x^2-2x+5)^2}\,dx
\end{aligned}$$

We have $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$ and $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}+c$. Make the substitution $u=x^2-2x+5$, $du=(2x-2)\,dx$ to get

$$\int \frac{(2x-2)\,dx}{x^2-2x+5} = \int \frac{du}{u} = \ln u + c = \ln(x^2-2x+5)+c$$

and

$$\int \frac{(2x-2)\,dx}{(x^2-2x+5)^2} = \int \frac{du}{u^2} = \frac{-1}{u}+c = \frac{-1}{x^2-2x+5}+c.$$

Make the substitution $2\tan\theta = x-1$, $2\sec\theta = \sqrt{x^2-2x+5}$, $2\sec^2\theta\,d\theta = dx$ to get

$$\int \frac{4\,dx}{x^2-2x+5} = \int \frac{4\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\tfrac{x-1}{2}\right) + c$$

and

$$\int \frac{16\,dx}{(x^2-2x+5)^2} = \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4}\,d\theta = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1+\cos 2\theta\;d\theta$$

$$= \theta + \tfrac{1}{2}\,\sin 2\theta + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\tfrac{x-1}{2}\right) + \tfrac{2(x-1)}{x^2-2x+5} + c\,.$$

Thus we have

$$I = \Bigg[-\ln x + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\frac{x-1}{2}$$

$$- \frac{1}{x^2-2x+5} - \tan^{-1}\frac{x-1}{2} - \frac{2(x-1)}{x^2-2x+5}\Bigg]_1^2$$

$$= \left[\ln\frac{x^2-2x+5}{x} + \frac{1}{x} - \frac{2x-1}{x^2-2x+5} + \tan^{-1}\frac{x-1}{2}\right]_1^2$$

$$= \left(\ln\tfrac{5}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tan^{-1}\tfrac{1}{2}\right) - \left(\ln 4 + 1 - \tfrac{1}{4}\right)$$

$$= \ln\tfrac{5}{8} - \tfrac{17}{20} + \tan^{-1}\tfrac{1}{2}\,.$$

2.41 Example: Find $\displaystyle\int \frac{\sec^3 x\,dx}{\sec x - 1}$.

Solution: Multiply the numerator and denominator by $\sec x + 1$ to get

$$\int \frac{\sec^3 x\,dx}{\sec x - 1} = \int \frac{\sec^3 x(\sec x + 1)}{(\sec^2 x - 1)}\,dx = \int \frac{\sec^4 x + \sec^3 x}{\tan^2 x}\,dx = \int \frac{\sec^4 x}{\tan^2 x}\,dx + \int \frac{\sec^3 x}{\tan^2 x}\,dx\,.$$

Make the substitution $u = \tan x$, $du = \sec^2 x\,dx$ to get

$$\int \frac{\sec^4 x}{\tan^2 x}\,dx = \int \frac{(\tan^2 x + 1)\sec^2 x\,dx}{\tan^2 x} = \int \frac{u^2+1}{u^2}\,du$$

$$= \int 1 + \frac{1}{u^2}\,du = u - \frac{1}{u} + c = \tan x - \cot x + c\,.$$

Make the substitution $v = \sin x$, $dv = \cos x\,dx$ and integrate by parts to get

$$\int \frac{\sec^3 x}{\tan^2 x}\,dx = \int \frac{dx}{\cos x\sin^2 x} = \int \frac{\cos x\,dx}{(1-\sin^2 x)\sin^2 x} = \int \frac{dv}{(1-v^2)\,v^2}$$

$$= \int \frac{1}{1-v^2} + \frac{1}{v^2}\;dv = \int \frac{\frac{1}{2}}{1-v} + \frac{\frac{1}{2}}{1+v} + \frac{1}{v^2}\;dv$$

$$= -\tfrac{1}{2}\ln|1-v| + \tfrac{1}{2}\ln|1+v| - \tfrac{1}{v} + c = \tfrac{1}{2}\ln\left|\tfrac{1+v}{1-v}\right| - \tfrac{1}{v} + c$$

$$= \tfrac{1}{2}\ln\tfrac{1+\sin x}{1-\sin x} - \csc x + c = \tfrac{1}{2}\ln\tfrac{(1+\sin x)^2}{(\cos x)^2} - \csc x + c = \ln\left|\tfrac{1+\sin x}{\cos x}\right| - \csc x + c\,.$$

Thus $\displaystyle\int \frac{\sec^3 x}{\sec x - 1}\,dx = \tan x - \cot x + \ln|\sec x + \tan x| - \csc x + c.$