Chapter 3. Approximate and Improper Integration

Approximate Integration

3.1 Definition: Let f be integrable on $[a,b]$. We can approximate the integral of f on $[a,b]$ by any Riemann sum

$$I = \int_a^b f(x)\,dx \cong \sum_{i=1}^n f(c_i)\Delta_i x$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta_i x = x_i - x_{i-1}$ and $c_i \in [x_{i-1}, x_i]$. The n^{th} **Left Endpoint Approximation** L_n, the n^{th} **Right Endpoint Approximation** R_n, and the n^{th} **Midpoint Approximation** M_n, for the integral $I = \int_a^b f(x)\,dx$ are the Riemann sums for f obtained by using the partition of $[a,b]$ into n equal sized subintervals and by choosing c_i to be the left endpoint, the right endpoint, or the midpoint of the i^{th} subinterval $[x_{i-1}, x_i]$. We have

$$L_n = \sum_{i=1}^n f(x_{i-1})\Delta x = \frac{b-a}{n}\Big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big)$$

$$R_n = \sum_{i=1}^n f(x_i)\Delta x = \frac{b-a}{n}\Big(f(x_1) + f(x_2) + \cdots + f(x_n)\Big)$$

$$M_n = \sum_{i=1}^n f\Big(\frac{x_{i-1}+x_i}{2}\Big)\Delta x = \frac{b-a}{n}\Big(f\left(\tfrac{x_0+x_1}{2}\right) + f\left(\tfrac{x_1+x_2}{2}\right) + \cdots + f\left(\tfrac{x_{n-1}+x_n}{2}\right)\Big)$$

where $x_i = a + \frac{b-a}{n}\,i$ and $\Delta x = \frac{b-a}{n}$.

3.2 Definition: Let f be integrable on $[a,b]$. The **Trapezoidal Approximation** T_n for the integral $I = \int_a^b f(x)\,dx$ is defined as follows. We use the partition of $[a,b]$ into n equal-sized subintervals, so we let $x_i = a + \frac{b-a}{n}\,i$ and $\Delta x = \frac{b-a}{n}$. Let g_i be the linear polynomial with $g_i(x_{i-1}) = f(x_{i-1})$ and $g_i(x_i) = f(x_i)$. Let g be the piecewise-linear function defined by $g(x) = g_i(x)$ for $x \in [x_{i-1}, x_i]$. We define

$$T_n = \int_a^b g(x)\,dx\,.$$

Note that

$$\int_{x_{i-1}}^{x_i} g(x)\,dx = \int_{x_{i-1}}^{x_i} g_i(x)\,dx = \frac{f(x_{i-1}) + f(x_i)}{2}\,\Delta x$$

(indeed, the integral measures the area of a trapezoid) so we have

$$\begin{aligned} T_n &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} g(x)\,dx = \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2}\,\Delta x = \frac{L_n + R_n}{2} \\ &= \frac{b-a}{2n}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big)\,. \end{aligned}$$

3.3 Definition: Let f be integrable on $[a,b]$. For an even positive integer n, we define the **Simpson Approximation** S_n for the integral $I = \int_a^b f(x)\,dx$ as follows. We partition $[a,b]$ into n equal-sized subintervals. Let $x_i = a + \frac{b-a}{n}\,i$ and $\Delta x = \frac{b-a}{n}$. For $k = 1, 2, \cdots, \frac{n}{2}$, let g_k be the quadratic polynomial with $g(x_{2k-2}) = f(x_{2k-2})$, $g(x_{2k-1}) = f(x_{2k-1})$ and $g(x_{2k}) = f(x_{2k})$. Let g be the piecewise-quadratic function given by $g(x) = g_k(x)$ for $x \in [x_{2k-2}, x_{2k}]$. We define

$$S_n = \int_a^b g(x)\,dx\,.$$

Note that if $h(x) = Ax^2 + Bx + C$ is the quadratic polynomial with $h(-1) = u$, $h(0) = v$ and $h(1) = w$, then we must have $u = h(-1) = A - B + C$, $v = h(0) = C$ and $w = h(1) = A + B + C$. Solving these three equations gives $A = \frac{u-2v+w}{2}$, $B = \frac{w-u}{2}$ and Cv so we have

$$\begin{aligned}\int_{-1}^1 h(x)\,dx &= \int_{-1}^1 \tfrac{u-2v+w}{2}\,x^2 + \tfrac{w-u}{2}\,x + v\;dx \\ &= \Big[\tfrac{u-2v+w}{6}\,x^3 + \tfrac{w-u}{4}\,x^2 + v\,x\Big]_{-1}^1 \\ &= \tfrac{u-2v+w}{3} + 2v = \tfrac{u+4v+w}{3}\,.\end{aligned}$$

It follows, by shifting and scaling, that

$$\int_{x_{2k-2}}^{x_{2k}} g_k(x)\,dx = \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\,\Delta x\,.$$

Thus

$$\begin{aligned}S_n &= \sum_{k=1}^{n/2} \int_{x_{2k-2}}^{x_{2k}} g(x)\,dx = \sum_{k=1}^{n/2} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\,\Delta x \\ &= \frac{b-a}{3n}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big)\,.\end{aligned}$$

3.4 Theorem: *(Error Bounds for Approximate Integration) Suppose that the higher order derivatives of f exist and are continuous on $[a,b]$. Let $I = \int_a^b f(x)\,dx$. and let L_n, R_n, T_n, M_n and S_n be the left endpoint, right endpoint, midpoint, trapezoidal and Simpson approximation of I. Then*

$$\begin{aligned}\big|L_n - I\big| &\le \frac{(b-a)^2}{2\,n} \max_{a\le x\le b} \big|f'(x)\big| \\ \big|R_n - I\big| &\le \frac{(b-a)^2}{2\,n} \max_{a\le x\le b} \big|f'(x)\big| \\ \big|T_n - I\big| &\le \frac{(b-a)^3}{12\,n^2} \max_{a\le x\le b} \big|f''(x)\big| \\ \big|M_n - I\big| &\le \frac{(b-a)^3}{24\,n^2} \max_{a\le x\le b} \big|f''(x)\big| \\ \big|S_n - I\big| &\le \frac{(b-a)^5}{180\,n^4} \max_{a\le x\le b} \big|f''''(x)\big|\end{aligned}$$

3.5 Example: Let $f(x)=\sin^2 x$. Find the exact value $I=\int_0^{4\pi/3} f(x)\,dx$, find the approximations L_8, R_8, M_8, T_8 and S_8, and find a bound on the error for each of these approximations.

Solution: The exact value of the integral is

$$I=\int_0^{4\pi/3}\sin^2 x\;dx=\int_0^{4\pi/3}\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\;dx=\Big[\tfrac{1}{2}\,x-\tfrac{1}{4}\sin 2x\Big]_0^{4\pi/3}=\tfrac{4\pi}{3}-\tfrac{\sqrt{3}}{8}\,.$$

When we divide the interval $[0,4\pi/3]$ into 8 equal subintervals, the size each of the subintervals is $\Delta x=\frac{\pi}{6}$ and the endpoints of the subintervals are $0,\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\frac{5\pi}{6},\pi,\frac{7\pi}{6},\frac{4\pi}{3}$. Thus the approximations are

$$\begin{aligned}
L_8&=\tfrac{\pi}{6}\Big(f(0)+f\big(\tfrac{\pi}{6}\big)+f\big(\tfrac{\pi}{3}\big)+f\big(\tfrac{\pi}{2}\big)+f\big(\tfrac{2\pi}{3}\big)+f\big(\tfrac{5\pi}{6}\big)+f(\pi)+f\big(\tfrac{7\pi}{6}\big)\Big)\\
&=\tfrac{\pi}{6}\big(0+\tfrac{1}{4}+\tfrac{3}{4}+1+\tfrac{3}{4}+\tfrac{1}{4}+0+\tfrac{1}{4}\big)=\tfrac{13\pi}{24}\,,\\
R_8&=\tfrac{\pi}{6}\Big(f\big(\tfrac{\pi}{6}\big)+f\big(\tfrac{\pi}{3}\big)+f\big(\tfrac{\pi}{2}\big)+f\big(\tfrac{2\pi}{3}\big)+f\big(\tfrac{5\pi}{6}\big)+f(\pi)+f\big(\tfrac{7\pi}{6}\big)+f\big(\tfrac{4\pi}{3}\big)\Big)\\
&=\tfrac{\pi}{6}\big(\tfrac{1}{4}+\tfrac{3}{4}+1+\tfrac{3}{4}+\tfrac{1}{4}+0+\tfrac{1}{4}+\tfrac{3}{4}\big)=\tfrac{2\pi}{3}\,,\\
T_8&=\tfrac{1}{2}\big(L_8+R_8\big)=\tfrac{29\pi}{48}\,,\\
M_8&=\tfrac{\pi}{6}\Big(f\big(\tfrac{\pi}{12}\big)+f\big(\tfrac{\pi}{4}\big)+f\big(\tfrac{5\pi}{12}\big)+f\big(\tfrac{7\pi}{12}\big)+f\big(\tfrac{3\pi}{4}\big)+f\big(\tfrac{11\pi}{12}\big)+f\big(\tfrac{13\pi}{12}\big)+f\big(\tfrac{15\pi}{12}\big)\Big)\\
&=\tfrac{\pi}{6}\Big(\tfrac{2-\sqrt{3}}{4}+\tfrac{1}{2}+\tfrac{2+\sqrt{3}}{4}+\tfrac{2+\sqrt{3}}{4}+\tfrac{1}{2}+\tfrac{2-\sqrt{3}}{4}+\tfrac{2-\sqrt{3}}{4}+\tfrac{1}{2}\Big)=\tfrac{\pi}{6}\Big(4-\tfrac{\sqrt{3}}{4}\Big)\,,\\
S_8&=\tfrac{\pi}{18}\Big(f(0)+4f\big(\tfrac{\pi}{6}\big)+2f\big(\tfrac{\pi}{3}\big)+4f\big(\tfrac{\pi}{2}\big)+2f\big(\tfrac{2\pi}{3}\big)+4f\big(\tfrac{5\pi}{6}\big)+2f(\pi)+4f\big(\tfrac{7\pi}{6}\big)+f\big(\tfrac{4\pi}{3}\big)\Big)\\
&=\tfrac{\pi}{18}\big(0+1+\tfrac{3}{2}+4+\tfrac{3}{2}+1+0+1+\tfrac{3}{4}\big)=\tfrac{43\pi}{72}\,.
\end{aligned}$$

Note that to find the values of f needed for the midpoint approximation M_8, we used the identity $f(x)=\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$. From this same identity, we obtain $f'(x)=\sin 2x$ and then $f''(x)=2\cos 2x$, $f'''(x)=-4\sin 2x$ and $f''''(x)=-8\cos 2x$. Thus we find that

$$\max_{0\le x\le 4\pi/3}\big|f'(x)\big|=1\,,\quad \max_{0\le x\le 4\pi/3}\big|f''(x)\big|=2\ \text{ and }\ \max_{0\le x\le 4\pi/3}\big|f''''(x)\big|=8\,.$$

The above theorem gives the following error bounds.

$$\begin{aligned}
\big|L_8-I\big|&\le\frac{(4\pi/3)^2}{16}\cdot 1=\tfrac{\pi^2}{9}\\
\big|R_r-I\big|&\le\frac{(4\pi/3)^2}{16}\cdot 1=\tfrac{\pi^2}{9}\\
\big|T_n-I\big|&\le\frac{(4\pi/3)^3}{12\cdot 6^2}\cdot 2=\tfrac{8\pi^3}{3^6}\\
\big|M_n-I\big|&\le\frac{(4\pi/3)^3}{24\cdot 6^2}\cdot 2=\tfrac{4\pi^3}{3^6}\\
\big|S_n-I\big|&\le\frac{(4\pi/3)^5}{180\cdot 6^4}\cdot 8=\tfrac{2^7\pi^5}{5\cdot 3^{11}}
\end{aligned}$$

Improper Integration

3.6 Definition: Suppose that $f:[a,b)\to\mathbb{R}$ is integrable on every closed interval contained in $[a,b)$. Then we define the **improper integral** of f on $[a,b)$ to be

$$\int_a^b f = \lim_{t\to b^-}\int_a^t f$$

provided the limit exists and, when the improper integral exists and is finite, we say that f is **improperly integrable** on $[a,b)$, (or that the improper integral of f on $[a,b)$ **converges**). In this definition we also allow the case that $b=\infty$, and then we have

$$\int_a^\infty f = \lim_{t\to\infty}\int_a^t f\,.$$

Similarly, if $f:(a,b]\to\mathbb{R}$ is integrable on every closed interval in $(a,b]$ then we define the **improper integral** of f on $(a,b]$ to be

$$\int_a^b f = \lim_{t\to a^+}\int_t^b f$$

provided the limit exists, and we say that f is **improperly integrable** on $(a,b]$ when the improper integral is finite. In this definition we also allow the case that $a=-\infty$. For a function $f:(a,b)\to\mathbb{R}$, which is integrable on every closed interval in (a,b), we choose a point $c\in(a,b)$, then we define the **improper integral** of f on (a,b) to be

$$\int_a^b f = \int_a^c f + \int_c^b f$$

provided that both of the improper integrals on the right exist and can be added, and we say that f is **improperly integrable** on (a,b) when both of the improper integrals on the right are finite. As an exercise, you should verify that the value of this integral does not depend on the choice of c.

3.7 Notation: For a function $F:(a,b)\to\mathbb{R}$ write

$$\Big[F(x)\Big]_{a^+}^{b^-} = \lim_{x\to b^-}F(x) - \lim_{x\to a^+}F(x)\,.$$

We use similar notation when $F:[a,b)\to\mathbb{R}$ and when $F:(a,b]\to\mathbb{R}$.

3.8 Note: Suppose that $f:(a,b)\to\mathbb{R}$ is integrable on every closed interval contained in (a,b) and that F is differentiable with $F'=f$ on (a,b). Then

$$\int_a^b f = \Big[F(x)\Big]_{a^+}^{b^-}\,.$$

A similar result holds for functions defined on half-open intervals $[a,b)$ and $(a,b]$.

Proof: Choose $c\in(a,b)$. By the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_a^b f &= \int_a^c f + \int_c^b f = \lim_{s\to a^+}\int_s^c f + \lim_{t\to b^-}\int_c^t f\\ &= \lim_{s\to a^+}\big(F(c)-F(s)\big) + \lim_{t\to b^-}\big(F(t)-F(c)\big)\\ &= \lim_{t\to b^-}F(t) - \lim_{s\to a^+}F(s) = \Big[F(x)\Big]_{a^+}^{b^-}\,.\end{aligned}$$

3.9 Example: Find $\int_0^1 \frac{dx}{x}$ and find $\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution: We have

$$\int_0^1 \frac{dx}{x} = \Big[\ln x\Big]_{0^+}^1 = 0 - (-\infty) = \infty$$

and

$$\int_0^1 \frac{dx}{\sqrt{x}} = \Big[2\sqrt{x}\Big]_{0^+}^1 = 2 - 0 = 2\,.$$

3.10 Example: Show that $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.

Solution: The case that $p = 1$ was dealt with in the previous example. If $p > 1$ so that $p - 1 > 0$ then we have

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_{0^+}^1 = \left(-\tfrac{1}{p-1}\right) - (-\infty) = \infty$$

and if $p < 1$ so that $1 - p > 0$ then we have

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{0^+}^1 = \left(\tfrac{1}{1-p}\right) - (0) = \tfrac{1}{1-p}\,.$$

3.11 Example: Show that $\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p > 1$.

Solution: When $p = 1$ we have

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{1}{x} = \Big[\ln x\Big]_1^\infty = \infty - 0 = \infty\,.$$

When $p > 1$ so that $p - 1 > 0$ we have

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^\infty = (0) - \left(-\tfrac{1}{p-1}\right) = \tfrac{1}{p-1}$$

and if $p < 1$ so that $1 - p > 0$ then we have

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = (\infty) - \left(\tfrac{1}{1-p}\right) = \infty\,.$$

3.12 Example: Find $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution: We have
$$\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty = 0-(-1)=1\,.$$

3.13 Example: Find $\displaystyle\int_0^1 \ln x\,dx$.

Solution: We have
$$\int_0^1 \ln x\;dx = \Big[x\ln x - x\Big]_{0^+}^1 = (-1)-(0) = -1\,,$$
since l'Hôpital's Rule gives $\displaystyle\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$

3.14 Theorem: *(Comparison) Let f and g be integrable on closed subintervals of (a,b), and suppose that $0\le f(x)\le g(x)$ for all $x\in(a,b)$. If g is improperly integrable on (a,b) then so is f and then we have*
$$\int_a^b f \le \int_a^b g\,.$$
On the other hand, if $\displaystyle\int_a^b f$ diverges then $\displaystyle\int_a^b g$ diverges, too. A similar result holds for functions f and g defined on half-open intervals.

Proof: The proof is left as an exercise.

3.15 Example: Determine whether $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges.

Solution: For $0\le x<\frac{\pi}{2}$ we have $\cos x\ge 1-\frac{2}{\pi}\,x$ so $\sec x\le \frac{1}{1-\frac{2}{\pi}\,x}$ hence $\sqrt{\sec x}\le\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}$. Let $u=1-\frac{2}{\pi}\,x$ so that $du=-\frac{2}{\pi}\,dx$. Then
$$\int_{x=0}^{\pi/2}\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}\,dx = \int_{u=1}^{0} -\tfrac{\pi}{2}\,u^{-1/2} = \Big[-\pi\,u^{1/2}\Big]_1^0 = \pi$$
which is finite. It follows that $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges, by comparison.

3.16 Example: Determine whether $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges.

Solution: For $0\le u$ we have $e^u\ge 1+u$, so for $0\le x$ we have $e^{x^2}\ge 1+x^2$, so $e^{-x^2}\le\dfrac{1}{1+x^2}$. Since
$$\int_0^\infty\frac{dx}{1+x^2} = \Big[\tan^{-1}x\Big]_0^\infty = \tfrac{\pi}{2}\,,$$
which is finite, we see that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges, by comparison.

3.17 Theorem: *(Estimation) Let f be integrable on closed subintervals of (a, b). If $|f|$ is improperly integrable on (a, b) then so is f, and then we have*

$$\left| \int_a^b f \right| \leq \int_a^b |f| \, .$$

A similar result holds for functions defined on half-open intervals.

Proof: The proof is left as an exercise.

3.18 Example: Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ converges.

Solution: We shall show that both of the integrals $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ and $\displaystyle\int_1^\infty \frac{\sin x}{x}\, dx$ converge.

Since $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, the function f defined by $f(0) = 1$ and $f(x) = \dfrac{\sin x}{x}$ for $x > 0$ is continuous (hence integrable) on $[0, 1]$. By part 1 of the Fundamental Theorem of Calculus, the function $\displaystyle\int_r^1 f(x)\, dx$ is a continuous function of r for $r \in [0, 1]$ and so we have

$$\int_0^1 \frac{\sin x}{x}\, dx = \lim_{r\to 0^+} \int_r^1 \frac{\sin x}{x}\, dx = \lim_{r\to 0^+} \int_r^1 f(x)\, dx = \int_0^1 f(x)\, dx \, ,$$

which is finite, so $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ converges.

Integrate by parts using $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\, dx$, $v = -\sin x$ and $dv = \cos x\, dx$ to get

$$\int_1^\infty \frac{\sin x}{x}\, dx = \left[-\frac{\cos x}{x} \right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2}\, dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\, dx \, .$$

Since $\left| \dfrac{\cos x}{x^2} \right| \leq \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, we see that $\displaystyle\int_1^\infty \left| \frac{\cos x}{x^2} \right| dx$ converges too, by comparison. Thus $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\, dx$ also converges by the Estimation Theorem.