Chapter 4. Applications of the Definite Integral

Area Between Curves

4.1 Note: Suppose that f and g are integrable on $[a,b]$ with $f(x) \le g(x)$ for all $x \in [a,b]$. We can approximate the area of the region R given by

$$a \le x \le b\ ,\ \ f(x) \le y \le g(x)$$

as follows. Choose a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a,b]$ and choose sample points $c_i \in [x_{i-1}, x_i]$. We divide the region R into strips with the i^{th} strip given by

$$x_{i-1} \le x \le x_i\ ,\ \ f(x) \le y \le g(x)\,.$$

The area $\Delta_i A$ of the i^{th} strip is approximately equal to the area of the rectangle with base $\Delta_i x = x_i - x_{i-1}$ and height $g(c_i) - f(c_i)$, that is

$$\Delta_i A \cong \big(g(c_i) - f(c_i)\big)\Delta_i x\,.$$

The area of the entire region R is

$$A = \sum_{i=1}^{n} \Delta_i A \cong \sum_{i=1}^{n} \big(g(c_i) - f(c_i)\big)\Delta_i x\,.$$

We notice that the sum on the right is a Riemann sum for the function $g(x) - f(x)$ on the interval $[a,b]$, so we define the exact area of the region R to be the limit of these Riemann sums.

4.2 Definition: Suppose that f and g are integrable on $[a,b]$ with $f(x) \le g(x)$ for all $x \in [a,b]$. We define the **area** of the region R given by

$$a \le x \le b\ ,\ \ f(x) \le y \le g(x)$$

to be

$$A = \int_a^b g(x) - f(x)\ dx\,.$$

4.3 Example: Find the area of the region R which lies between the x-axis and the parabola $y = 1 - x^2$.

Solution: The region R is given by $-1 \le x \le 1$, $0 \le y \le 1 - x^2$ and so the area is

$$A = \int_{-1}^{1} 1 - x^2\ dx = \Big[x - \tfrac{1}{3}\,x^3\Big]_{-1}^{1} = \big(1 - \tfrac{1}{3}\big) - \big(-1 + \tfrac{1}{3}\big) = \tfrac{4}{3}\,.$$

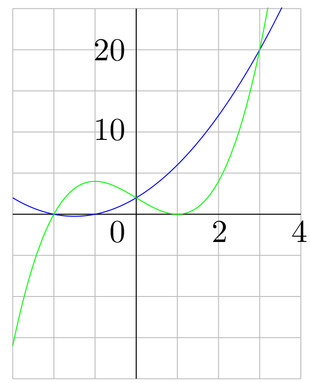
4.4 Example: Find the area of the region R which lies between the curves $y = x^2 + 3x + 2$ and $y = x^3 - 3x + 2$.

Solution: Let $f(x) = x^2 + 3x + 2$ and $g(x) = x^3 - 3x + 2$. First, let us find the points of intersection of the two curves and determine where $f(x) \geq g(x)$. We have

$$f(x) - g(x) = (x^2 + 3x + 2) - (x^3 - 3x + 2) = -(x^3 - x^2 - 6x) = -x(x-3)(x+2)$$

and so $f(x) = g(x)$ when $x \in \{-2, 0, 3\}$ with $f(x) \geq g(x)$ for $x \in (-\infty, -2] \cup [0, 3]$ and $f(x) \leq g(x)$ for $x \in [-2, 0] \cup [3, \infty)$. Next we make a table of values and sketch the curves. The curve $y = f(x)$ is shown in blue and the curve $y = g(x)$ is shown in green.

x	$f(x)$	$g(x)$
-3	2	-16
-2	0	0
-1	0	4
0	2	2
1	6	0
2	12	4
3	20	20



The region R consists of two parts with the first part given by $-2 \leq x \leq 0$, $f(x) \leq y \leq g(x)$ and the second part given by $0 \leq x \leq 3$, $g(x) \leq y \leq f(x)$ and so the total area is

$$\begin{aligned} A &= \int_{-2}^{0} g(x) - f(x)\ dx + \int_{0}^{3} f(x) - g(x)\ dx \\ &= \int_{-2}^{0} x^3 - x^2 - 6x\ dx + \int_{0}^{3} -x^3 + x^2 + 6x\ dx \\ &= \Big[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 3x^2\Big]_{-2}^{0} + \Big[-\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 + 3x^2\Big]_{0}^{3} \\ &= -\big(4 + \tfrac{8}{3} - 12\big) + \big(-\tfrac{81}{4} + 9 + 27\big) = \tfrac{16}{3} + \tfrac{63}{4} = \tfrac{253}{12}\,. \end{aligned}$$

4.5 Example: Find the area of a circle of radius r.

Solution: The area of a circle of radius r is equal to 4 times the area of the region given by $0 \leq x \leq r$, $0 \leq y \leq \sqrt{r^2 - x^2}$, and so the area of the circle is

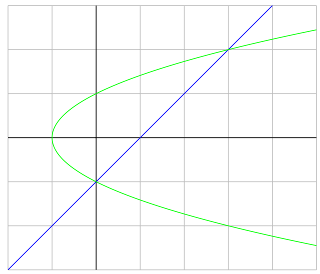
$$A = 4\int_0^r \sqrt{r^2 - x^2}\, dx\,.$$

To solve the integral, we make the substitution $r\sin\theta = x$ so that $r\cos\theta = \sqrt{r^2 - x^2}$ and $r\cos\theta\, d\theta = dx$ to get

$$\begin{aligned} A &= \int_{x=0}^{r} 4\sqrt{r^2 - x^2}\, dx = \int_{\theta=0}^{\pi/2} 4r\cos\theta \cdot r\cos\theta\ d\theta = \int_0^{\frac{\pi}{2}} 4r^2\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)\, d\theta \\ &= r^2\int_0^{\pi/2} 2 - 2\cos 2\theta\ d\theta = r^2\Big[2\theta - \sin 2\theta\Big]_0^{\pi/2} = r^2(\pi - 0) = \pi r^2\,. \end{aligned}$$

4.6 Example: Find the area of the region R which lies between the curves $y = x - 1$ and $y^2 = x + 1$.

Solution: The line $y = x - 1$ is shown in blue and the parabola $y^2 = x + 1$ is shown in green.



We now find the area in two ways. For the first solution, we divide the given region along the y-axis into two regions with the first given by $-1 \le x \le 0$, $-\sqrt{x+1} \le y \le \sqrt{x+1}$, and the second given by $0 \le x \le 3$, $x - 1 \le y \le \sqrt{x+1}$. The area of the first region is

$$A_1 = \int_{-1}^{0} \sqrt{x+1} - \left(-\sqrt{x+1}\right)\,dx = \int_{-1}^{0} 2(x+1)^{1/2}\,dx = \left[\tfrac{4}{3}(x+1)^{3/2}\right]_{-1}^{0} = \tfrac{4}{3}\,,$$

and the area of the second region is

$$A_2 = \int_{0}^{3} \sqrt{x+1} - (x-1)\,dx = \left[\tfrac{2}{3}(x+1)^{3/2} - \tfrac{1}{2}\,x^2 + x\right]_{0}^{3} = \left(\tfrac{16}{3} - \tfrac{9}{2} + 3\right) - \left(\tfrac{2}{3}\right) = \tfrac{19}{6}\,,$$

and so the total area is

$$A = A_1 + A_2 = \tfrac{4}{3} + \tfrac{19}{6} = \tfrac{27}{6} = \tfrac{9}{2}\,.$$

For the second solution we shall interchange the roles of x and y, thinking of x as a function of y. The line is given by $x = y + 1$ and the parabola is given by $x = y^2 - 1$, so the region R is given by $-1 \le y \le 2$, $y^2 - 1 \le x \le y + 1$, and hence the area is

$$\begin{aligned} A &= \int_{y=-1}^{2} (y+1) - (y^2 - 1)\,dy = \int_{-1}^{2} -y^2 + y + 2\,dy = \left[-\tfrac{1}{3}\,y^3 + \tfrac{1}{2}\,y^2 + 2y\right]_{-1}^{2} \\ &= \left(-\tfrac{8}{3} + 2 + 4\right) - \left(\tfrac{1}{3} + \tfrac{1}{2} - 2\right) = \tfrac{9}{2}\,. \end{aligned}$$

Volume by Cross-Section

4.7 Note: Suppose that a solid S lies in space between $x = a$ and $x = b$, and its cross-sectional area at x (that is the area of the intersection of the solid with the plane perpendicular to the x-axis at the position x) is equal to $A(x)$, where A is integrable on $[a, b]$. We can approximate the volume of S as follows. Choose a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ and choose sample points $c_i \in [x_{i-1}, x_i]$. Divide the solid into strips where the i^{th} strip lies between $x = x_{i-1}$ and $x = x_i$ and has thickness $\Delta_i x = x_i - x_{i-1}$. The volume of the i^{th} strip is

$$\Delta_i V \cong A(c_i)\Delta_i x$$

and the total volume of S is

$$V = \sum_{i=1}^{n} \Delta_i V \cong \sum_{i=1}^{n} A(c_i)\Delta_i x\,.$$

We notice that the sum on the right is a Riemann sum for the function $A(x)$ on $[a, b]$, so we define the exact volume of S to be the limit of these Riemann sums.

4.8 Definition: Suppose that a solid S lies in space between $x = a$ and $x = b$, and that its cross-sectional area at x is equal to $A(x)$, where A is integrable on $[a, b]$. We define the **volume** of S to be

$$V = \int_a^b A(x)\,dx\,.$$

4.9 Example: Let f and g be integrable on $[a, b]$ with $0 \le f(x) \le g(x)$ for all $x \in [a, b]$. Let R be the region in the xy-plane given by

$$a \le x \le b\ ,\ \ f(x) \le y \le g(x)$$

and let S be the solid obtained by revolving R about the y-axis. Then the cross-section of S at position x is an annulus (that is the region between two concentric circles) with inner radius $f(x)$ and outer radius $g(x)$, so the cross-sectional area is

$$A(x) = \pi g(x)^2 - \pi f(x)^2\,.$$

Thus the volume of S is

$$V = \int_a^b \pi\big(g(x)^2 - f(x)^2\big)\ dx\,.$$

4.10 Example: Find the volume of a cone of base radius r and height h.

Solution: Such a cone (lying on its side) can be obtained by revolving the triangular region R given by $0 \le x \le h$, $0 \le y \le \frac{r}{h}\,x$ about the x-axis, so the volume is

$$V = \int_0^h \pi\left(\tfrac{r}{h}\,x\right)^2\ dx = \tfrac{\pi\,r^2}{h^2}\left[\tfrac{1}{3}\,x^3\right]_0^h = \tfrac{\pi\,r^2}{h^2}\left(\tfrac{1}{3}\,h^3\right) = \tfrac{1}{3}\,\pi\,r^2 h\,.$$

4.11 Example: Find the volume of a sphere of radius r.

Solution: One half of such a sphere can be obtained by revolving the region R given by $0 \le x \le r$, $0 \le y \le \sqrt{r^2 - x^2}$ about the x-axis, and so the volume is

$$V = 2\int_0^r \pi(r^2 - x^2)\,dx = 2\pi\Big[r^2 x - \tfrac{1}{3}\,x^3\Big]_0^r = 2\pi\left(r^3 - \tfrac{1}{3}\,r^3\right) = \tfrac{4}{3}\,\pi\,r^3\,.$$

4.12 Example: Find the volume of the football-shaped solid S which is obtained by revolving the region R which lies under one arch of the sine curve about the x-axis.

Solution: The region R is given by $0 \le x \le \pi$, $0 \le y \le \sin x$ and so the volume is

$$V = \int_0^\pi \pi \sin^2 x \, dx = \int_0^\pi \pi \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx = \pi \Big[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x \Big]_0^\pi = \tfrac{\pi^2}{2} \, .$$

4.13 Example: Let R be the (infinitely long) region given by $1 \le x$, $0 \le y \le \frac{1}{x}$, and let S be the solid obtained by revolving R about the x-axis.

Solution: The area of the region R is

$$A = \int_1^\infty \frac{1}{x} \, dx = \Big[\ln x \Big]_1^\infty = \infty$$

because $\lim\limits_{x \to \infty} \ln x = \infty$. The volume of the solid, on the other hand, is

$$V = \int_1^\infty \pi \cdot \frac{1}{x^2} \, dx = \pi \left[-\frac{1}{x} \right]_1^\infty = \pi$$

because $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

4.14 Remark: The above example gives rise to the following amusing paradox. It would require an infinite amount of paint to cover the region R but only a finite amount of paint to fill the solid S. But surely if we fill S with paint we have also covered R with paint! The resolution to this paradox lies in the fact that our calculation holds for mathematical paint which can flow down into an arbitrarily small tube.

4.15 Example: Find the volume of the solid S given by $x^2 + y^2 \le r^2$, $x^2 + z^2 \le r^2$ (this is the intersection of two cylinders).

Solution: To find the cross-section at x (where $-r \le x \le r$) we treat x as a fixed constant, and then the cross-section is given by $y^2 \le r^2 - x^2$ and $z^2 \le r^2 - x^2$, or equivalently by $|y| \le \sqrt{r^2 - x^2}$ and $|z| \le \sqrt{r^2 - x^2}$. Thus we see (somewhat surprisingly) that the cross-section at x is the square given by $|y| \le \sqrt{r^2 - x^2}$ and $|z| \le \sqrt{r^2 - x^2}$. This square has sides of length $2\sqrt{r^2 - x^2}$ so the cross-sectional area is

$$A(x) = 4(r^2 - x^2) \, .$$

Thus the volume of S is

$$V = \int_{-r}^{r} A(x) \, dx = \int_{-r}^{r} 4(r^2 - x^2) \, dx = 4 \Big[r^2 x - \tfrac{1}{3} x^3 \Big]_{-r}^{r} = 4 \left(\tfrac{2}{3} r^3 - \left(-\tfrac{2}{3} r^3 \right) \right) = \tfrac{16}{3} r^3 \, .$$

Volume by Cylindrical Shells

4.16 Note: Suppose that f and g are integrable on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$, let R be the region in the xy-plane given by

$$a \leq x \leq b\ ,\ f(x) \leq y \leq g(x)\,,$$

and let S be the solid obtained by revolving R about the y-axis. We can approximate the volume of S as follows. We choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, and we choose intermediate points $c_i \in [x_{i-1}, x_i]$. We divide the region into strips where the i^{th} strip R_i is given by $x_{i-1} \leq x \leq x_i$, $f(x) \leq y \leq g(x)$. We divide the solid into "cylindrical shells" where the i^{th} shell S_i is obtained by revolving R_i about the y-axis. The volume of the i^{th} shell is

$$\Delta_i V \cong 2\pi\, c_i \big(g(c_i) - f(c_i)\big) \Delta_i x$$

where $\Delta_i x = x_i - x_{i-1}$. The total volume of S is

$$S = \sum_{i=1}^{n} \Delta_i V \cong \sum_{i=1}^{n} 2\pi\, c_i \big(g(c_i) - f(c_i)\big) \Delta_i x\,.$$

The sum on the right is a Riemann sum for the function $2\pi\, x\big(g(x) - f(x)\big)$ on $[a,b]$.

4.17 Definition: Suppose that f and g are integrable on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$, let R be the region in the xy-plane given by

$$a \leq x \leq b\ ,\ f(x) \leq y \leq g(x)\,,$$

and let S be the solid obtained by revolving R about the y-axis. We define the **volume** of S to be

$$V = \int_a^b 2\pi\, x\big(f(x) - g(x)\big)\, dx\,.$$

4.18 Example: Find the volume of a sphere of radius r.

Solution: One half of such a sphere can be obtained by revolving the region R given by $0 \leq x \leq r$, $0 \leq y \leq \sqrt{r^2 - x^2}$ about the y-axis, so the volume is

$$V = 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\ dx\,.$$

To solve the integral, we let $u = r^2 - x^2$ so that $du = -2x\,dx$, and we have

$$V = \int_0^r 4\pi x\sqrt{r^2 - x^2}\, dx = \int_{r^2}^0 -2\pi\,\sqrt{u}\ du = \Big[\tfrac{4}{3}\,\pi\, u^{3/2}\Big]_0^r = \tfrac{4}{3}\,\pi\, r^3\,.$$

4.19 Example: Find the volume of the discus-shaped solid S obtained by revolving the region R given by $0 \leq x \leq \frac{\pi}{2}$, $-\cos x \leq y \leq \cos x$ about the y-axis.

Solution: The volume is

$$V = \int_0^{\pi/2} 2\pi\, x\big(\cos x - (-\cos x)\big)\, dx = \int_0^{\pi/2} 4\pi\, x\cos x\ dx\,.$$

To solve the integral, we integrate by parts using $u = x$ and $dv = \cos x\, dx$ to get

$$V = 4\pi\Big[x\sin x - \int \sin x\ dx\Big]_0^{\pi/2} = 4\pi\Big[x\sin x + \cos x\Big]_0^{\pi/2} = 4\pi\big(\tfrac{\pi}{2} - 1\big) = 2\pi^2 - 4\pi\,.$$

4.20 Example: A bowl is in the shape of the surface obtained by revolving the part of the parabola $y = x^2$ with $0 \le x \le 2$ about the y-axis. Find the capacity of the bowl.

Solution: The capacity of the bowl is the volume of the liquid in the bowl when it is full. The liquid is in the shape of the solid S obtained by revolving the region R given by $0 \le x \le 2$, $x^2 \le y \le 4$ about the y-axis. We find the volume in two ways. Using the method of cylindrical shells, we have

$$V = \int_{x=0}^{2} 2\pi\, x(4 - x^2)\, dx = \pi \int_0^2 8x - 2x^3\ dx = \pi\Big[4x^2 - \tfrac{1}{2}\, x^4\Big]_0^2 = 8\pi\,.$$

For the second solution, we interchange the roles of x and y. Note that the region R is also given by $0 \le y \le 4, 0 \le x \le \sqrt{y}$. Using the method of cross-sections we obtain

$$V = \int_{y=0}^{4} \pi\big(\sqrt{y}\big)^2\, dy = \pi \int_0^4 y\, dy = \pi\Big[\tfrac{1}{2}\, y^2\Big]_0^4 = 8\pi\,.$$

4.21 Example: Find the volume of the solid torus (that is the doughnut-shaped solid) S with inner radius $R - r$ and outer radius $R + r$, where $0 < r < R$.

Solution: Note that such a torus can be obtained by revolving the disc D given by

$$(x - R)^2 + y^2 \le r^2$$

about the y-axis. We find the volume in two ways. First we use the method of cylindrical shells. The disc D is given by

$$R - r \le x \le R + r \ \text{ and } \ -\sqrt{r^2 - (x-R)^2} \le y \le \sqrt{r^2 - (x-R)^2}$$

so the volume of the taurus is

$$V = \int_{x=R-r}^{R+r} 2\pi\, x \cdot 2\sqrt{r^2 - (x-R)^2}\ dx\,.$$

To solve this integral we let $r\sin\theta = (x - R)$ so that $r\cos\theta = \sqrt{r^2 - (x-R)^2}$ and $r\cos\theta\, d\theta = dx$ to get

$$\begin{aligned} V &= \int_{x=R-r}^{R+r} 4\pi\, x\sqrt{r^2 - (x-R)^2}\ dx = \int_{\theta=-\pi/2}^{\pi/2} 4\pi \cdot (R + r\sin\theta)\cdot r\cos\theta \cdot r\cos\theta\, d\theta \\ &= 4\pi\, r^2 \int_{-\pi/2}^{\pi/2} R\cos^2\theta + r\sin\theta\cos^2\theta\ d\theta = 4\pi\, r^2 \int_{-\pi/2}^{\pi/2} R\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) + r\sin\theta\cos^2\theta\ d\theta \\ &= 4\pi\, r^2\Big[R\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + r \cdot \tfrac{1}{3}\cos^3\theta\Big]_{-\pi/2}^{\pi/2} = 4\pi\, r^2 \cdot R\left(\tfrac{\pi}{4} + \tfrac{\pi}{4}\right) = 2\pi^2\, r^2 R\,. \end{aligned}$$

For the second solution, we interchange the roles of x and y and use the cross-section method. The disc D is given by

$$-r \le y \le r \ \text{ and } \ R - \sqrt{r^2 - y^2} \le x \le R + \sqrt{r^2 - y^2}$$

and so the volume is

$$\begin{aligned} V &= \int_{y=-r}^{r} \pi\Big((R + \sqrt{r^2 - y^2})^2 - (R - \sqrt{r^2 - y^2})^2\Big)\, dy = \int_{-r}^{r} 4\pi R\sqrt{r^2 - y^2}\ dy \\ &= 4\pi R \cdot \tfrac{1}{2}\pi r^2 = 2\pi^2 r^2 R\,. \end{aligned}$$

We used the fact that $\displaystyle\int_{-r}^{r} \sqrt{r^2 - y^2}\ dy$ measures the area of a semicircle.

Arclength

4.22 Note: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve $y=f(x)$ with $a\le x\le b$. We approximate the length of C as follows. Choose a partition $a=x_0<x_1<\cdots<x_n=b$ of $[a,b]$. Write $\Delta_i x = x_i - x_{i-1}$ and $\Delta_i y = f(x_i)-f(x_{i-1})$. By the Mean Value Theorem, we can choose sample points $c_i\in[x_{i-1},x_i]$ so that

$$f'(c_i)=\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}=\frac{\Delta_i y}{\Delta_i x}\,.$$

Let C_i be the part of the curve $y=f(x)$ with $x_{i-1}\le x\le x_i$, and let D_i be the line segment from $\big(x_{i-1},f(x_{i-1})\big)$ to $\big(x_i,f(x_i)\big)$. The length $\Delta_i L$ of C_i is approximately equal to the length of D_i, that is

$$\Delta_i L\cong\sqrt{(\Delta_i x)^2+(\Delta_i y)^2}=\sqrt{1+\left(\frac{\Delta_i y}{\Delta_i x}\right)^2}\cdot\Delta_i x=\sqrt{1+f'(c_i)^2}\cdot\Delta_i x$$

and so the total length of C is

$$L=\sum_{i=1}^n L_i\cong\sum_{i=1}^n\sqrt{1+f'(c_i)^2}\cdot\Delta_i x\,.$$

The sum on the right is a Riemann sum for the function $\sqrt{1+f'(x)^2}$ on $[a,b]$.

4.23 Definition: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). We define the **length** (or the **arclength**) of the curve $y=f(x)$ from $x=a$ to $x=b$ to be

$$L=\int_a^b\sqrt{1+f'(x)^2}\,dx\,.$$

We say that f is **rectifiable** when the length L is finite.

4.24 Example: Find the length of the curve $y=x^2$ with $0\le x\le 2$.

Solution: Let $f(x)=x^2$ so $f'(x)=2x$. The length of the curve is

$$L=\int_0^2\sqrt{1+f'(x)^2}\,dx=\int_0^2\sqrt{1+4x^2}\,dx\,.$$

To solve the integral, let $\tan\theta=2x$ so $\sec\theta=\sqrt{1+4x^2}$ and $\sec^2\theta\,d\theta=2\,dx$ to get

$$\begin{aligned}\int\sqrt{1+4x^2}\,dx&=\int\tfrac12\sec^3\theta\,d\theta=\tfrac14\sec\theta\tan\theta+\tfrac14\ln\big|\sec\theta+\tan\theta\big|+c\\&=\tfrac12\,x\sqrt{1+4x^2}+\tfrac14\ln\big|2x+\sqrt{1+4x^2}\big|+c\end{aligned}$$

so that

$$L=\int_0^2\sqrt{1+4x^2}\,dx=\Big[\tfrac12\,x\sqrt{1+4x^2}+\tfrac14\ln\big(2x+\sqrt{1+4x^2}\big)\Big]_0^2=\sqrt{17}+\tfrac14\ln\big(4+\sqrt{17}\big)\,.$$

Surface Area

4.25 Note: The area of (the lateral surface of) a cone of base radius r and slant height l is given by $A = \pi r l$. More generally, the area of a slice of a cone with base radius r, top radius s, and slant height l, is given by

$$A = \pi(r+s)\,l\,.$$

4.26 Note: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve in the xy-plane given by $y = f(x)$ with $a \leq x \leq b$. Let S be the surface obtained by revolving C about the x-axis. We can approximate the area of the surface S as follows. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$. Write $\Delta_i x = x_i - x_{i-1}$ and $\Delta_i y = f(x_i) - f(x_{i-1})$. Use the Mean Value Theorem to select $c_i \in [x_{i-1}, x_i]$ so that $f'(c_i) = \dfrac{\Delta_i y}{\Delta_i x}$. Let C_i be the part of the curve C with $x_{i-1} \leq x \leq x_i$, and let S_i denote the slice of the surface S which is obtained by revolving C_i about the x-axis. Let D_i be the line segment from $\big(x_{i-1}, f(x_{i-1})\big)$ to $\big(x_i, f(x_i)\big)$ and let T_i be the slice of a cone obtained by revolving D_i about the x-axis. The area $\Delta_i A$ of the slice S_i is approximately equal to the area of T_i, that is

$$\begin{aligned}\Delta_i A &\cong \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta_i L \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{1 + f'(c_i)^2}\cdot \Delta_i x \\ &\cong 2\pi f(c_i)\sqrt{1 + f'(c_i)^2}\cdot \Delta_i x\,.\end{aligned}$$

The final sum is a Riemann sum for $2\pi f(x)\sqrt{1+f'(x)^2}$ on $[a,b]$.

4.27 Definition: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve given by $y = f(x)$ with $a \leq x \leq b$. Let S be the surface obtained by revolving C about the x-axis. We define the **area** of the surface S to be

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\;dx\,.$$

4.28 Note: A similar argument to the one given above shows that we can approximate the area of the surface S obtained by revolving the curve C given by $y = f(x)$ with $a \leq x \leq b$ about the y-axis by a Riemann sum for the function $2\pi\,x\sqrt{1+f'(x)^2}$ on $[a,b]$.

4.29 Definition: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve given by $y = f(x)$ with $a \leq x \leq b$. Let S be the surface obtained by revolving C about the y-axis. We define the **area** of the surface S to be

$$A = \int_a^b 2\pi\,x\sqrt{1+f'(x)^2}\;dx\,.$$

4.30 Example: Find the area of a sphere of radius r.

Solution: Such a sphere can be obtained by revolving the portion of the curve $y=\sqrt{r^2-x^2}$ with $-r\le x\le r$ about the x-axis. Let $f(x)=\sqrt{r^2-x^2}$ so that $f'(x)=-\dfrac{x}{\sqrt{r^2-x^2}}$ and

$$\sqrt{1+f'(x)^2}=\sqrt{1+\frac{x^2}{r^2-x^2}}=\sqrt{\frac{r^2}{r^2-x^2}}=\frac{r}{\sqrt{r^2-x^2}}\,.$$

Using the first of the above two definitions, the surface area is

$$A=\int_{-r}^{r}2\pi\,f(x)\sqrt{1+f'(x)^2}\;dx=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}\;dx=\int_{-r}^{r}2\pi\,r\;dx=4\pi\,r^2\,.$$

Alternatively, we can obtain half of such a sphere by revolving the curve $y=\sqrt{r^2-x^2}$ with $0\le x\le r$ about the y-axis. Using the second of the above two definitions, the area of the sphere is

$$A=2\int_{x=0}^{r}2\pi\,x\sqrt{1+f'(x)^2}\;dx=\int_{0}^{r}4\pi\,x\cdot\frac{r}{\sqrt{r^2-x^2}}\;dx\,.$$

To solve the integral, we let $u=r^2-x^2$ so $du=-2x\,dx$ to get

$$A=\int_{x=0}^{r}\frac{4\pi r\,x}{\sqrt{r^2-x^2}}\,dx=\int_{u=r^2}^{0}-2\pi\,r\,u^{-1/2}\;du=\Big[-4\pi\,r\,u^{1/2}\Big]_{r^2}^{0}=4\pi\,r^2\,.$$

4.31 Example: Find the area of a torus of inner radius $R-r$ and outer radius $R+r$.

Solution: Half of such a torus can be obtained by revolving the curve $y=\sqrt{r^2-(R-x)^2}$ with $R-r\le x\le R+r$ about the y-axis. Let $f(x)=\sqrt{r^2-(x-R)^2}$. Then we have

$$f'(x)=\frac{-(x-R)}{\sqrt{r^2-(x-R)^2}}$$

so

$$\sqrt{1+f'(x)^2}=\sqrt{1+\frac{(x-R)^2}{r^2-(x-R)^2}}=\sqrt{\frac{r^2}{r^2-(x-R)^2}}=\frac{r}{\sqrt{r^2-(x-R)^2}}$$

and so, using the second of the above two definitions, the surface area is

$$A=2\int_{R-r}^{R+r}2\pi\,x\cdot\frac{r}{\sqrt{r^2-(x-R)^2}}\;dx=4\pi\,r\int_{R-r}^{R-r}\frac{x\,dx}{\sqrt{r^2-(x-R)^2}}\,.$$

To solve the integral, make the substitution $r\sin\theta=x-R$ so $r\cos\theta=\sqrt{r^2-(x-R)^2}$ and $r\cos\theta\,d\theta=dx$. Then we obtain

$$A=4\pi\,r\int_{x=R-r}^{R+r}\frac{x\,dx}{\sqrt{r^2-(x-R)^2}}=4\pi\,r\int_{\theta=-\pi/2}^{\pi/2}\frac{(R+r\sin\theta)\cdot r\cos\theta\,d\theta}{r\cos\theta}$$

$$=4\pi\,r\int_{-\pi/2}^{\pi/2}R+r\sin\theta\;d\theta=4\pi\,r\Big[R\theta-r\cos\theta\Big]_{-\pi/2}^{\pi/2}=4\,\pi^2r\,R\,.$$

Density

4.32 Example: Suppose a rod lies along the x-axis from $x = a$ to $x = b$, and the linear density (that is mass per unit length) of the rod is equal to $\rho(x)$, where $\rho(x)$ is integrable on $[a.b]$. We can approximate the mass of the rod as follows. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ and choose sample points $c_i \in [x_{i-1}, x_i]$. The mass of the part of the rod between $x = x_{i-1}$ and $x = x_i$ is

$$\Delta_i M \cong \rho(c_i)\Delta_i x$$

and so the total mass of the rod is

$$M = \sum_{i=1}^{n} \Delta_i M \cong \sum_{i=1}^{n} \rho(c_i)\Delta_i x\,.$$

The sum on the right is a Riemann sum for the function $\rho(x)$. The exact mass of the rod is the limit of these Riemann sums, that is

$$M = \int_a^b \rho(x)\,dx\,.$$

4.33 Example: Suppose that a ball of radius R has varying density, and the density at each point which lies at a distance of r units from the origin is equal to $\rho(r)$, where we suppose that ρ is integrable on $[0, R]$. We can approximate the mass of the ball as follows. Choose a partition $0 = r_0 < r_1 < \cdots < r_n = R$ of the interval $[0, R]$, and choose sample points $c_i \in [r_{i-1}, r_i]$. Divide the sphere into spherical shells using concentric spheres of radius r_i. The volume of the i^{th} spherical shell is $\Delta_i V \cong 4\pi {c_i}^2 \Delta_i r$ so its mass is

$$\Delta_i M \cong \rho(c_i)\Delta_i V \cong 4\pi {c_i}^2 \rho(c_i)\Delta_i r\,.$$

The total mass of the ball is

$$M = \sum_{i=1}^{n} \Delta_i M \cong \sum_{i=1}^{n} 4\pi {c_i}^2 \rho(c_i)\Delta_i r\,.$$

This is a Riemann sum, and the exact mass of the ball is the limit of these Riemann sums, that is

$$M = \int_a^b 4\pi r^2 \rho(r)\ dr\,.$$

Force

4.34 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \le x \le 2$, $-5 \le z \le 5$ together with the two ends given by $-2 \le x \le 2$, $x^2 \le y \le 4$ with $z = \pm 5$ (where the y-axis is pointing upwards). The tank is filled with a liquid of density ρ. The pressure $P(h)$ (force per unit area) exerted by the liquid on each wall at all points which lie at a depth h is given by

$$P = \rho g h$$

where g is the gravitational constant. Find the total force exerted by the liquid on each of the ends of the tank.

Solution: We provide a less formal solution than we gave in previous examples. Although we make no mention of a Riemann sum, it should be apparent that we are in fact approximating the total force by a Riemann sum and then calculating the exact force as a limit of Riemann sums. Along one of the ends of the tank, consider a thin horizontal slice at position y of thickness Δy. The slice is at a depth of $h = 4 - y$ so the pressure at all points is $P = \rho g h = \rho g(4 - y)$. The width of the slice is equal to $2\sqrt{y}$, so the area of the slice is $\Delta A = 2\sqrt{y}\,\Delta y$, and so the force exerted by the water on the slice is

$$\Delta F = P\,\Delta A = \rho g(4 - y) \cdot 2\sqrt{y}\,\Delta y\,.$$

The total force exerted on the end of the tank is

$$\begin{aligned} F &= \int_{y=0}^{4} \rho g(4 - y) \cdot 2\sqrt{y}\;dy \; = \rho g \int_0^4 8\,y^{1/2} - 2\,y^{3/2}\,dy \\ &= \rho g \Big[\tfrac{16}{3}\,y^{3/2} - \tfrac{4}{5}\,y^{5/2}\Big]_0^4 = \rho g\left(\tfrac{128}{3} - \tfrac{128}{5}\right) = \tfrac{256}{15}\,\rho g\,. \end{aligned}$$

4.35 Example: A charged rod, of charge Q (with its charge evenly distributed along its length) lies along the x-axis from $x = 0$ to $x = 2$. A small object of charge q lies at position $(x, y) = (2, 1)$. Find the force exerted by the rod on the object. Use the fact that the force exerted by one small object of charge q_1 at position p_1 on another of charge q_2 at position p_2 is equal to

$$F = \frac{k\,q_1 q_2}{|u|^2} \cdot \frac{u}{|u|}$$

where k is a constant and u is the direction vector from p_1 to p_2, that is $u = p_2 - p_1$.

Solution: Again, we provide a less formal solution, making no mention of Riemann sums. Consider a small slice of rod at position x of thickness Δx. Since the rod has length 2, the charge per unit length is $\frac{Q}{2}$ and so the charge on the slice of rod is $\Delta Q = \frac{Q}{2}\,\Delta x$. The distance from the slice, which is at position $(x, 0)$ to the small object, which is at position $(2, 1)$, is equal to $r = |u| = \sqrt{(2 - x)^2 + 1}$ and so the magnitude of the force exerted by the slice on the object is

$$\Delta F = \frac{kq \cdot \frac{Q}{2}\,\Delta x}{(2 - x)^2 + 1}\,.$$

By similar triangles, the x and y-components of the force, exerted by the slice of rod on the object, are given by

$$\Delta F_x = \frac{2-x}{\sqrt{(2-x)^2+1}} \cdot \Delta F = \frac{kqQ\,(2-x)\,\Delta x}{2\left((2-x)^2+1\right)^{3/2}}$$
$$\Delta F_y = \frac{1}{\sqrt{(2-x)^2+1}}\,\Delta F = \frac{kqQ\,\Delta x}{2\left((2-x)^2+1\right)^{3/2}}\,.$$

and so the x-component of the total force is

$$F_x = \int_{x=0}^{2} \frac{kqQ\,(2-x)\,dx}{2\left((2-x)^2+1\right)^{3/2}}\,.$$

To solve the integral, we let $u = (2-x)^2+1$ so that $du = -2(2-x)\,dx$ to get

$$F_x = \int_{x=0}^{2} \frac{kqQ\,(2-x)\,dx}{2\left((2-x)^2+1\right)^{3/2}} = \int_{u=5}^{1} \frac{-kqQ\cdot\frac{1}{2}\,du}{2\,u^{3/2}} = kqQ\int_{5}^{1} -\tfrac{1}{4}\,u^{-3/2}\ du$$
$$= kqQ\left[\tfrac{1}{2}\,u^{-1/2}\right]_5^1 = \tfrac{1}{2}\,kqQ\big(1-\tfrac{1}{\sqrt{5}}\big)\,.$$

The y-component of the total force is

$$F_y = \int_{x=0}^{2} \frac{kqQ\ dx}{2\left((2-x)^2+1\right)^{3/2}}\,.$$

To solve this integral, we let $\tan\theta = 2-x$ so $\sec\theta = \sqrt{(2-x)^2+1}$ and $\sec^2\theta\,d\theta = -dx$. Then

$$\int \frac{kqQ\ dx}{2\left((2-x)^2+1\right)^{3/2}} = \int \frac{-kqQ\sec^2\theta\,d\theta}{2\sec^3\theta} = \int -\tfrac{1}{2}\,kqQ\cos\theta\ d\theta$$
$$= -\tfrac{1}{2}\,kqQ\sin\theta + c = -\tfrac{1}{2}\,kqQ\cdot\tfrac{2-x}{\sqrt{(2-x)^2+1}} + c$$

and so

$$F_y = \int_{x=0}^{2} \frac{kqQ\ dx}{2\left((2-x)^2+1\right)^{3/2}} = \left[-\tfrac{1}{2}\,kqQ\cdot\tfrac{2-x}{\sqrt{(2-x)^2+1}}\right]_0^2 = \tfrac{1}{2}\,kqQ\cdot\tfrac{2}{\sqrt{5}}\,.$$

Thus the total force exerted by the rod on the object, expressed as a vector, is

$$\big(F_x, F_y\big) = \tfrac{1}{2}\,kqQ\Big(1-\tfrac{1}{\sqrt{5}}\,,\ \tfrac{2}{\sqrt{5}}\Big)\,.$$

4.36 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \le x \le 2$, $-5 \le z \le 5$ together with the two ends given by $-2 \le x \le 2$, $x^2 \le y \le 4$ with $z = \pm 5$ (where the y-axis is pointing vertically). The tank is filled with a liquid of density ρ. Find the work required to pump all the liquid out of the tank, bringing it all to the level of the top of the tank. Use the fact that the work required to raise a small object of mass m from height h_1 to height h_2 is equal to

$$W = mgh$$

where $h = h_2 - h_1$.

Solution: We provide an informal solution. Consider a thin slice of liquid at position y of thickness Δy. The slice is in the shape of a thin rectangle of length $l = 10$, width $w = 2\sqrt{y}$ and thickness Δy, so its volume is $\Delta V = 20\sqrt{y}\,\Delta y$, and so its mass is given by $\Delta M = \rho\Delta V = 20\rho\sqrt{y}\,\Delta y$. All the water in this slice must be raised from height $h_1 = 4-y$ to height $h_2 = 4$, and so the work done in pumping the water in this slice is

$$\Delta W = gh\,\Delta M = 20\rho g\,(4-y)\sqrt{y}\,\Delta y\,.$$

The total work required to pump all the water in the tank is

$$\begin{aligned} W &= \int_{y=0}^{4} 20\rho g\,(4-y)\sqrt{y}\;dy = 20\rho g\int_0^4 4\,y^{1/2} - y^{3/2}\;dy \\ &= 20\rho g\Big[\tfrac{8}{3}\,y^{3/2} - \tfrac{2}{5}\,y^{5/2}\Big]_0^4 = 20\rho g\big(\tfrac{64}{3} - \tfrac{64}{5}\big) = \tfrac{2560}{3}\,\rho g\,. \end{aligned}$$

4.37 Example: A chain, of length π and mass M, lies along the x-axis. Find the work required to lift the chain and lie it along the top half of the circle $x^2 + (y-1)^2 = 1$ (where the y-axis points upwards).

Solution: Let θ be as shown below. For a thin slice of the chain (when it is lying on the top half of the circle) at position θ of thickness $\Delta\theta$, the mass of the slice is $\Delta M = \frac{M}{\pi}\Delta\theta$, and the height of the slice above the x-axis is $y = 1 + \sin\theta$, so the work done in lifting the slice from the x-axis is $\Delta W = gy\,\Delta M = \frac{gM}{\pi}(1+\sin\theta)\Delta\theta$. The total work is

$$W = \int_{\theta=0}^{\theta=\pi} \tfrac{gM}{\pi}(1+\sin\theta)\,d\theta = \tfrac{gM}{\pi}\Big[\theta - \cos\theta\Big]_{\theta=0}^{\pi} = \tfrac{gM}{\pi}(\pi+2)\,.$$