Chapter 5. Introduction to Differential Equations

Differential Equations

5.1 Definition: An (ordinary) **differential equation**, or **DE**, is an equation which involves a function, say $y = y(x)$, of a single variable x, along with some of its derivatives $y'(x)$, $y''(x)$, etc. The **order** of a DE is the highest of the orders of the derivatives which occur in the equation. For example, the equation $y''(x) + 2y'(x)y(x)^3 = \sin x$ is a second order DE.

A **solution** to a DE is a function $y = y(x)$ which makes the equation true for all x in some interval. A DE can have many solutions. To solve a DE you must find the **general solution**, which means to find all possible solutions. Often, the general solution will involve arbitrary constants and the number of arbitrary constants will be equal to the order of the DE.

Sometimes we require that a solution to a DE satisfies one or more additional conditions, called **initial conditions**. A DE together with an initial condition (or a set of initial conditions) is called an **initial value problem**, or an **IVP**. Often, in an IVP, the number of initial conditions is equal to the order of the DE, and there is exactly one solution.

5.2 Example: Find a solution of the form $y = ax^2 + bx + c$ to the DE $y''y' + x^2 = y$.

Solution: Let $y = ax^2 + bx + c$. Then $y' = 2ax + b$ and $y'' = 2a$ and so we have $y''y' + x^2 = y \iff 2a(2ax+b) + x^2 = ax^2 + bx + c \iff x^2 + 4a^2\,x + 2ab = ax^2 + bx + c$. Equating coefficients gives $1 = a$, $4a^2 = b$ and $2ab = c$, and so we must have $a = 1$, $b = 4$ and $c = 8$. Thus the only such solution is $y = x^2 + 4x + 8$.

5.3 Example: Find two distinct constants r_1 and r_2 such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\,e^{r_1 x} + b\,e^{r_2 x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

Solution: Let $y = e^{rx}$. Then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$ and so $y'' + 3y' + 2y = 0 \iff r^2\,e^{rx} + 3r\,e^{rx} + 2\,e^{rx} = 0 \iff (r^2+3r+2)e^{rx} = 0 \iff (r+1)(r+2)\,e^{rx} = 0 \iff r = -1$ or $r = -2$. Thus we can take $r_1 = -1$ and $r_2 = -2$.

Now, let $y = a\,e^{r_1 x} + b\,e^{r_2 x} = a\,e^{-x} + b\,e^{-2x}$. Then $y' = -a\,e^{-x} - 2b\,e^{-2x}$ and $y'' = a\,e^{-x} + 4b\,e^{-2x}$ and so we have

$$y'' + 3y' + 2y = a\,e^{-x} + 4b\,e^{-2x} - 3a\,e^{-x} - 6b\,e^{-2x} + 2a\,e^{-x} + 2b\,e^{-2x} = 0\,.$$

This shows that $y = a\,e^{-x} + b\,e^{-2x}$ is a solution to the DE. Also, note that $y(0) = a + b$ and $y'(0) = -a - 2b$, and so to get $y(0) = 1$ and $y'(0) = 0$ we need $a + b = 1$ and $-a - 2b = 0$. Solve these two equations to get $a = 2$ and $b = -1$. Thus the required solution is $y = 2\,e^{-x} - e^{-2x}$.

5.4 Example: A rock is thrown downwards at 5 m/s from the top of a 100 m cliff and it falls to the ground. Assuming that the rock accelerates downwards at 10 m/s^2, find the speed of the rock when it lands.

Solution: Let $x(t)$ be the height of the rock, in meters, after t seconds. We must solve the IVP which consists of the 2$^{\text{nd}}$ order DE $x''(t) = -10$ and the two initial conditions $x'(t) = -5$ and $x(0) = 100$. We have

$$\begin{aligned} x''(t) &= -10 \\ \int x''(t)\,dt &= \int -10\,dt \\ x'(t) &= -10t + c_1 \end{aligned}$$

where c_1 is a constant. Since $x'(0) = -5$ we find that $c_1 = -5$, so we have

$$\begin{aligned} x'(t) &= -10t - 5 \\ \int x'(t)\,dt &= \int -10t - 5\,dt \\ x(t) &= -5t^2 - 5t + c_2 \end{aligned}$$

where c_2 is another constant. Since $x(0) = 100$ we have $c_2 = 100$ and so the solution to the IVP is $x(t) = -5t^2 - 5t + 100$. To find out when the rock lands, we solve $x(t) = 0$:

$$\begin{aligned} 0 &= -5t^2 - 5t + 100 \\ 0 &= t^2 + t - 20 \\ &= (t+5)(t-4) \end{aligned}$$

so it lands when $t = 4$. Since $x'(4) = -45$, the rock lands at a speed of 45 m/s.

Direction Fields

5.5 Definition: The graph of a solution $y = y(x)$ to a DE is called a **solution curve**.

5.6 Note: It is easy to sketch the solution curves to any DE of the form

$$y'(x) = F(x, y(x))$$

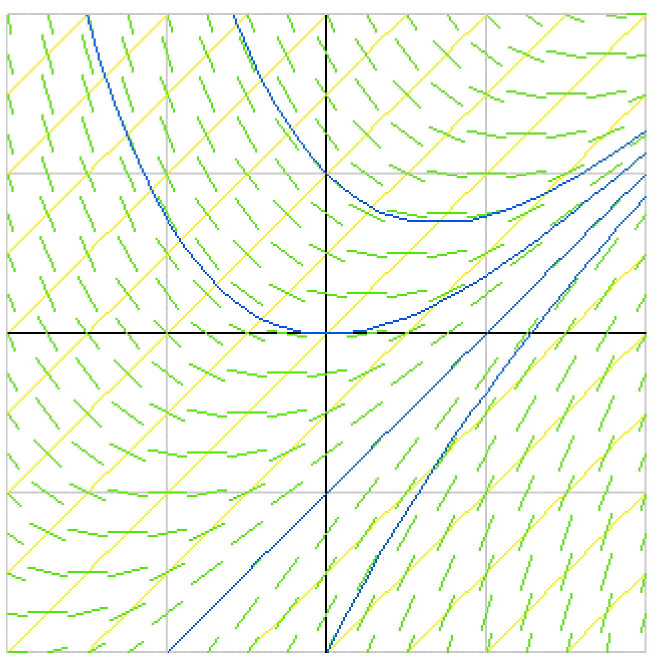
in the following way. First choose many points (x, y), and for each point (x, y) find the value of $F(x, y)$. If $y = y(x)$ is any solution to the DE, so that $y'(x) = F(x, y)$, then $F(x, y)$ is the slope of the solution curve at the point (x, y). At each point (x, y), draw a short line segment with slope $F(x, y)$. The resulting picture is called the **slope field** or the **direction field** of the DE. If we choose enough points (x, y) it should be possible to visualize the solution curves; they follow the direction of the short line segments.

To draw the direction field of the DE $y'(x) = F(x, y)$ by hand, it helps to first lightly draw several **isoclines**; these are the curves $F(x, y) = m$, where m is a constant. Along the isocline $F(x, y) = m$ we then draw many short line segments of slope m.

To draw the graph of the solution to the IVP $y'(x) = F(x, y)$, with $y(x_0) = y_0$, sketch the direction field for the DE $y'(x) = F(x, y)$ and then draw the solution curve which passes through the point (x_0, y_0).

5.7 Example: Sketch the direction field for the DE $y' = x - y$, then sketch the solution curves through each of the points $(x_0, y_0) = (0, -2)$, $(0, -1)$, $(0, 0)$ and $(0, 1)$.

Solution: The isoclines are the lines $x - y = m$. To sketch the direction field, we first lightly draw the lines $x - y = m$ for several values of m. These are shown below in yellow for $m = -\frac{7}{2}, -\frac{6}{2}, -\frac{5}{2}, \cdots, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}$. Then, along each isocline, we draw many short line segments of the appropriate slope; on the isocline $x - y = m$ we draw line segments of slope m. These are shown in green. The solution curves through each of the points $(x_0, y_0) = (0, -2)$, $(0, -1)$, $(0, 0)$ and $(0, 1)$ are shown below in blue.



5.8 Note: We can approximate the solution to the IVP $y'(x) = F\big(x, y(x)\big)$ with $y(a) = b$ using the following method, which is known as **Euler's Method**. Pick a small value Δx, which we call the **step size**. Let $x_0 = a$ and $y_0 = b$. Having found x_n and y_n, we let

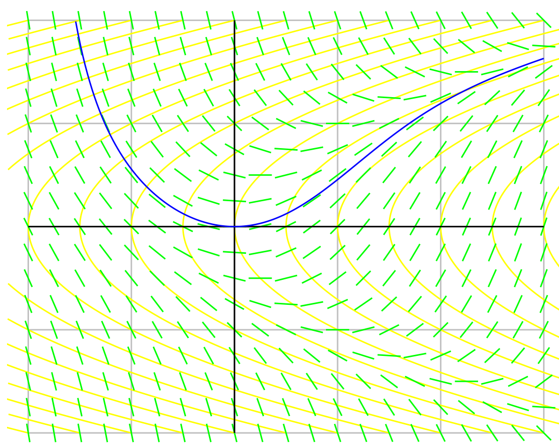
$$\begin{aligned} x_{n+1} &= x_n + \Delta x \\ y_{n+1} &= y_n + F(x_n, y_n)\,\Delta x\,. \end{aligned}$$

The solution curve $y = f(x)$ is then approximated for values $x \geq a$ by the piecewise linear curve whose graph has vertices at the points (x_n, y_n). Note that the slope of the line segment from (x_n, y_n) to (x_{n+1}, y_{n+1}) is equal to the slope of the direction field at the point (x_n, y_n). If we also wish to approximate the solution for values $x \leq a$, we can construct points (x_n, y_n) with $n < 0$ by letting

$$\begin{aligned} x_{n-1} &= x_n - \Delta x \\ y_{n-1} &= y_n - F(x_n, y_n)\,\Delta x\,. \end{aligned}$$

5.9 Example: Consider the IVP $y' = x - y^2$ with $y(0) = 0$. Sketch the direction field for the given DE along with the graph of the solution curve $y = f(x)$. With the help of a calculator, apply Euler's method with step size $\Delta x = \frac{1}{2}$ to approximate the value of $f(3)$.

Solution: The isocline (curve of constant slope) $y' = m$ is the sideways parabola $m = x - y^2$, or $x = y^2 + m$. The isoclines are shown in yellow, the slope field is shown in green, and the solution curve with $y(0) = 0$ is shown in blue.



We let $x_0 = 0$ and $y_0 = 0$. For $k \geq 0$ we set $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x, y) = x - y^2$. We make a table listing the values of x_k, y_k and $F(x_k, y_k)$.

k	x_k	y_k	$F(x_k, y_k) = x_k - {y_k}^2$
0	0	0	0
1	0.5	0	0.5
2	1.0	0.25	0.9375
3	1.5	0.71875	0.9833984375
4	2.0	1.210449219	0.534812688
5	2.5	1.477855563	0.315942935
6	3.0	1.635827030	

Thus we have $f(3) \cong y_6 \cong 1.6$.

Separable First Order Equations

5.10 Definition: A **separable** first order DE is a DE which can be written in the form

$$f\big(y(x)\big)\, y'(x) = g(x)\,.$$

for some continuous functions $f(y)$ and $g(x)$.

5.11 Note: $y = y(x)$ is a solution to the separable DE $f(y)\, y' = g(x)$ when

$$\int f(y(x))\, y'(x)\, dx = \int g(x)\, dx\,,$$

and by the change of variables formula, we have $\displaystyle\int f(y(x))\, y'(x)\, dx = \int f(y)\, dy$. So to solve the DE, we rewrite it as $f(y)\, dy = g(x)\, dx$ and then integrate both sides.

5.12 Example: Solve the DE $y' = x^2 y$.

Solution: We write the DE as $\dfrac{dy}{y} = x^2\, dx$, assuming $y \neq 0$, and integrate both sides to get

$$\begin{aligned} \ln|y| &= \tfrac{1}{3}x^3 + c \\ |y| &= e^{\frac{1}{3}x + c} \\ y &= \pm e^c e^{x^3/3} = Ae^{x^3/3}\,, \end{aligned}$$

where c is an arbitrary constant, and we set $A = \pm e^c$, so A is an arbitrary non-zero constant. Notice that $y = 0$ is also a solution to the DE, so the general solution is $y = Ae^{x^3/3}$, where A is an arbitrary constant.

5.13 Example: Solve the IVP $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$ with $y(1) = \pi$.

Solution: We rewrite the DE as $(2y + \cos y)dy = 6x^2\, dx$ then integrate both sides to get

$$\begin{aligned} \int (2y + \cos y)dy &= \int 6x^2\, dx \\ y^2 + \sin y &= 2x^3 + c \end{aligned}$$

where c is any constant. In this example we cannot solve for y explicitly as a function of x. To find the solution which satisfies the initial condition $y(1) = \pi$, we substitute $x = 1$ and $y = \pi$ into the above implicit solution to get $\pi^2 + \sin\pi = 2 + c$ so we find that $c = \pi^2 - 2$. Thus the solution to the IVP $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$, $y(1) = \pi$ is given implicitly by

$$y^2 + \sin y = 2x^3 + \pi^2 - 2\,.$$

Linear First Order Equations

5.14 Definition: A **linear** first order DE is a DE which can be written in the form

$$y'(x) + p(x)\,y(x) = q(x)$$

for some continuous functions $p(x)$ and $q(x)$.

5.15 Note: There is a trick which can be used to solve the linear DE $y' + py = q$. The trick is to find a function $\lambda = \lambda(x)$, called an **integrating factor**, such that $\lambda' = \lambda\,p$ so that $(\lambda\,y)' = \lambda\,y' + \lambda' y = \lambda\,y' + \lambda\,p\,y$. If we can find λ then we can solve the DE as follows:

$$\begin{aligned} y' + p\,y &= q \\ \lambda\,y' + \lambda\,p\,y &= \lambda\,q \\ (\lambda\,y)' &= \lambda\,q \\ \lambda\,y &= \int \lambda\,q\,dx \\ y &= \frac{1}{\lambda}\int \lambda\,q\,dx \end{aligned}$$

To find an integrating factor λ we must find a solution to the DE $\lambda'(x) = \lambda(x)p(x)$. This is a separable DE, so we rewrite it as $(1/\lambda)d\lambda = p(x)\,dx$ and integrate both sides

$$\begin{aligned} \int \frac{d\lambda}{\lambda} &= \int p(x)\,dx \\ \ln|\lambda| &= \int p(x)\,dx \\ |\lambda| &= e^{\int p(x)\,dx} \\ \lambda &= \pm e^{\int p(x)\,dx} \end{aligned}$$

Since any integrating factor will do , we can take $\lambda = e^{\int p(x)\,dx}$, and when we solve the integral $\int p(x)\,dx$ it is not necessary to keep track of the constant of integration. We summarize this in the following theorem.

5.16 Theorem: *The general solution to the linear DE $y'(x) + p(x)y(x) + q(x) = 0$ is*

$$y(x) = \frac{1}{\lambda(x)}\int \lambda(x)q(x)\,dx\ ,\ \text{where } \lambda(x) = e^{\int p(x)\,dx}\,.$$

5.17 Example: Find the general solution to the DE $y' - x^2 y = 0$.

Solution: This DE is both separable and linear. We already found the solution to this DE in the previous section by treating it at a separable DE. Now we will solve it again, using our method for solving linear DEs. An integrating factor is $\lambda = e^{\int -x^2\,dx} = e^{-x^3/3}$, and the solution is $y = \dfrac{1}{\lambda}\displaystyle\int 0\,dx = e^{x^3/3}\,c$, where c is a constant.

5.18 Example: Find the solution to the IVP $y' + 2y = e^{-5x}$, $y(0) = 1$.

Solution: An integrating factor is $\lambda = e^{\int 2\,dx} = e^{2x}$, and the solution to the DE is

$$y = \frac{1}{\lambda}\int \lambda\, e^{-5x}\,dx = e^{-2x}\int e^{-3x}\,dx = e^{-2x}\big(-\tfrac{1}{3}e^{-3x} + c\big) = -\tfrac{1}{3}e^{-5x} + c\,e^{-2x}\,,$$

where c is an arbitrary constant. Since $y(0) = 1$, we have $1 = -\frac{1}{3} + c$ and so $c = \frac{4}{3}$. Thus the solution to the IVP is $y = \frac{4}{3}e^{-2x} - \frac{1}{3}e^{-5x}$.

5.19 Example: Find the solution to the IVP $y' - 2xy = x$, $y(0) = 0$.

Solution: An integrating factor is $\lambda = e^{\int -2x\,dx} = e^{-x^2}$, and the solution to the DE is

$$y = \frac{1}{\lambda}\int x\,\lambda\,dx = e^{x^2}\int x\,e^{-x^2}\,dx = e^{x^2}\big(-\tfrac{1}{2}e^{-x^2} + c\big) = c\,e^{x^2} - \tfrac{1}{2}\,.$$

Since $y(0) = 0$ we have $c = \frac{1}{2}$ so the solution to the IVP is $y = \frac{1}{2}(e^{x^2} - 1)$.

5.20 Example: Solve the IVP $xy'' + y' = 4x$ with $y(1) = y(2) = 1$.

Solution: Write $u = y'$ so that $u' = y''$. Then the DE can be written as $xu' + u = 4x$. This is linear since we can write it as $u' + \frac{1}{x}\,u = 4$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$, and the solution is

$$u = \frac{1}{x}\int 4x\,dx = \frac{1}{x}(2x^2 + a) = 2x + \frac{a}{x}$$

where a is a constant, that is $y' = 2x + \dfrac{a}{x}$. Thus

$$y = \int 2x + \frac{a}{x}\,dx = x^2 + a\ln x + b$$

where b is a constant. To get $y(1) = 1$ we need $1 + b = 1$ so $b = 0$, and to get $y(2) = 1$ we need $4 + a\ln 2 = 1$ so $a = -\frac{3}{\ln 2}$. Thus the solution is

$$y = x^2 - \frac{3\ln x}{\ln 2} = x^2 - 3\log_2 x\,.$$

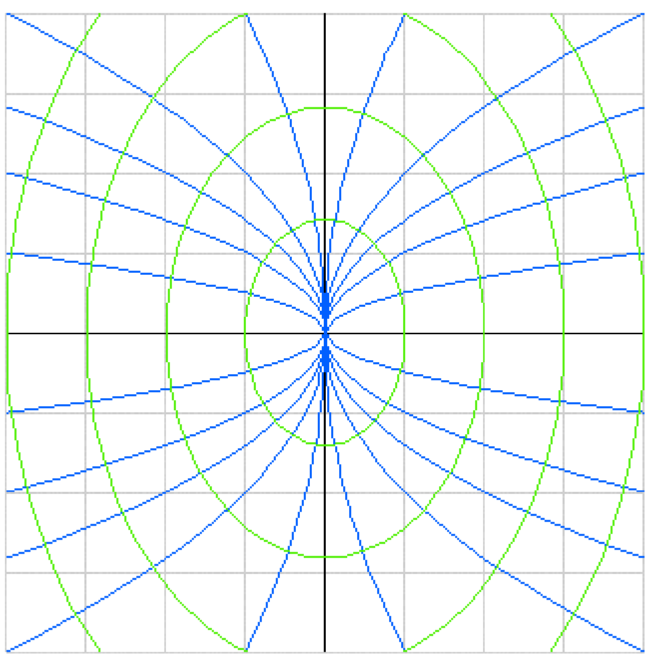
5.21 Definition: An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally. For example, each straight line $y = mx$ through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = r^2$, where r can be any positive constant.

5.22 Example: Find the orthogonal trajectories of the family of parabolas $x = k\,y^2$, where k is an arbitrary constant.

Solution: Differentiating $x = k\,y^2$ we obtain $1 = 2ky\,y'$ so the parabola $x = k\,y^2$ has slope $y' = \dfrac{1}{2ky}$ at each point. Since $k = \dfrac{x}{y^2}$, the parabola has slope $y' = \dfrac{1}{2ky} = \dfrac{y}{2x}$. Since the orthogonal trajectories are perpendicular to the parabolas, their slope is $y' = -\dfrac{2x}{y}$. So to find the orthogonal trajectories, we solve the DE $y' = -\dfrac{2x}{y}$. This is a separable DE, so we rewrite it as $y\,dy = -2x\,dx$ and integrate:

$$\begin{aligned}\int y\,dy &= \int -2x\,dx\\ \frac{1}{2}y^2 &= -x^2 + c\\ x^2+\frac{y^2}{2} &= c\end{aligned}$$

Thus the orthogonal trajectories are the ellipses $x^2 + \dfrac{y^2}{2} = c$, where c is an arbitrary positive constant. Some of the parabolas and ellipses in these families are shown below.



5.23 Definition: A quantity $y = y(t)$ is said to **grow or decay exponentially** if it satisfies the DE $y'(t) = k\,y(t)$ for some constant k. This DE is both separable and linear. Let us solve it as a linear DE (as an exercise, try solving it as a separable DE). An integrating factor is $\lambda = e^{\int -k\,dt} = e^{-kt}$, and the general solution is

$$y = \frac{1}{\lambda}\int \lambda \cdot 0\,dt = e^{kt}\int 0\,dt = c\,e^{kt}$$

where c is an arbitrary constant. Notice that c is equal to $y(0)$, so the solution is

$$y(t) = y(0)\,e^{kt}\,.$$

When $f(0)$ and k are positive, we say that y **grows exponentially**. When $f(0) > 0$ and $k < 0$ we say that y **decays exponentially**.

5.24 Example: Suppose that a bacteria culture grows exponentially. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000. Find a formula for the number of bacteria after t hours, and determine when the count was 3,000.

Solution: Let $y(t)$ be the count after t hours. Since the count grows exponentially, we have $y(t) = c\,e^{kt}$ for some positive constants c and k. Since $y(2) = 600$ we have $c\,e^{2k} = 600$, and since $y(8) = 75,000$ we have $c\,e^{8k} = 75,000$. We divide these two equations to get

$$\begin{aligned}\frac{c\,e^{8k}}{c\,e^{2k}} &= \frac{75,000}{6,000}\\ e^{6k} &= 125\\ e^{2k} &= 5\end{aligned}$$

so $k = \frac{1}{2}\ln 5$. Since $c\,e^{2k} = 600$ we have $5c = 600$ so $c = 120$. Thus $y(t) = 120e^{(\frac{1}{2}\ln 5)t} = 120(5^{t/2})$. The count was 3,000 when $120 \cdot 5^{t/2} = 3000$. Solve this to find that $t = 4$.

5.25 Example: A certain proportion of the carbon in all living plant material is the radioactive isotope C^{14}. It is believed that this proportion has not changed in the last several hundred thousand years. After a plant dies, the amount of C^{14} decays exponentially. The **half-life** of C^{14} is about 5730 years, which means that after 5730 years, one half of the initial C^{14} will be left.

Suppose that a parchment is found which contains 70% as much C^{14} as it did initially. Determine the age of the parchment.

Solution: Let $y(t)$ be the amount of C^{14} remaining in the parchment after t years. Since it decays exponentially, we have $y(t) = y(0)\,e^{kt}$. Since $y(5730) = \frac{1}{2}y(0)$, we have

$$\begin{aligned}y(0)\,e^{5730\,k} &= \frac{y(0)}{2}\\ e^{5730\,k} &= \frac{1}{2}\\ 5730\,k &= -\ln 2\\ k &= -\,\frac{\ln 2}{5730}\,.\end{aligned}$$

We want to find the value of t such that $y(t) = .70\,y(0)$, so we solve for t:

$$\begin{aligned} y(0)e^{kt} &= .7\,y(0) \\ e^{kt} &= .7 \\ kt &= \ln(.7) \\ t = \frac{\ln(.7)}{k} &= -\frac{5730\,\ln(.7)}{\ln 2} \cong 2950\,. \end{aligned}$$

Thus the parchment is about 2950 years old.

5.26 Note: Newton's Law of Cooling (or Warming) states that the rate of cooling (or warming) of an object is proportional to the temperature difference between the object and its surroundings. That is, if $T(t)$ is the temperature of the object at time t, and if K is the constant temperature of the surroundings, then

$$T'(t) = k(K - T)$$

for some constant k.

5.27 Example: A glass of water is taken from the refrigerator, where the temperature is 4°, and placed on a table, where the temperature is 20°. After 6 minutes, the water is found to be 11°. Find the temperature of the water after another 3 minutes.

Solution: Let $T(t)$ be the temperature of the water after t minutes. Then $T'(t) = k(20-T)$ for some constant k. This is a linear DE since we can write it as $T' + kT = 20k$ (its also separable). An integrating factor is $\lambda(t) = e^{\int k\,dt} = e^{kt}$, and the general solution is

$$T = \frac{1}{\lambda(t)}\int 20k\,\lambda(t)\,dt = e^{-kt}\int 20k\,e^{kt}\,dt = e^{-kt}(20e^{kt} + c) = 20 + c\,e^{kt}\,,$$

where c is a constant. Since $T(0) = 4$ we have $4 = 20+c$ so $c = -16$. Thus $T(t) = 20-16e^{kt}$. Since $T(6) = 11$ we have $11 = 20 - 16e^{6k}$, so $e^{6k} = \frac{9}{16}$, and $k = \frac{1}{6}\ln(\frac{9}{16})$. Thus

$$T(t) = 20 - 16e^{(\frac{1}{6}\ln\frac{9}{16})t} = 20 - 16\left(\tfrac{9}{16}\right)^{t/6} = 20 - 16\left(\tfrac{3}{4}\right)^{t/3},$$

and so $T(9) = 20 - 16\left(\frac{3}{4}\right)^3 = 20 - \frac{27}{4} = \frac{53}{4}$. The temperature after 9 minutes is 13.25°.

5.28 Example: In a simple electric circuit with a battery, producing a voltage of E volts, a resistor, of resistence R ohms, and an inductor, with an inductance of L henries, the current $I(t)$ at time t satisfies the DE

$$L\,I'(t) + R\,I(t) = E\,.$$

Given that $E = 12$, $R = 4$, $L = 2$ and $I(0) = 0$, find $I(t)$.

Solution: This DE is quite similar to the one that appears in Newton's Law of Cooling (or Warming). Put in the given values for E, R and L into the given DE to get $2\,I' + 4\,I = 12$. This is linear since we can write it as $I' + 2\,I = 6$. An integrating factor is $\lambda = e^{\int 2\,dt} = e^{2t}$ and the general solution is

$$I = e^{-2t}\int 6\,e^{2t}\,dt = e^{-2t}\big(3e^{2t} + c\big) = 3 + c\,e^{-2t}\,.$$

Put in $I(0) = 0$ to get $3 + c = 0$ so that $c = -3$, and so we have $I(t) = 3 - 3\,e^{-2t}$.

5.29 Note: In a typical **mixing problem**, a solution containing a given concentration c_1 of some substance (maybe salt in water) enters a tank at a fixed rate r_1. The mixture is kept stirred, and it is drained at another rate r_2. The problem is to find the amount $y(t)$ of substance in the tank at time t. We solve it by solving the DE $y'(t) = r_1c_1 - r_2c_2$, where c_2 is the concentration of the substance in the tank; that is $c_2 = \dfrac{y}{V}$, where V is the volume of the solution in the tank. If r_1 and r_2 are constant, then the volume is $V = V(0) + (r_1 - r_2)t$. In general, V satisfies the DE $V'(t) = r_1(t) - r_2(t)$.

5.30 Example: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine, at a concentration of .03 kg/L, enters that tank at a rate of 25 L/min. The solution is kept well mixed and drains from the tank at the same rate. Find the amount of salt in the tank after 5 hours.

Solution: Let $y(t)$ be the amount of salt in the tank, in kilograms, after t minutes. We must solve the IVP $y' = r_1c_1 - r_2c_2$, with $y(0) = 20$, where $r_1 = r_2 = 25$, $c_1 = .03$, and $c_2 = \dfrac{y}{V} = \dfrac{y}{5000}$. The DE becomes $y' = (25)(.03) - (25)\dfrac{y}{5000} = .75 - \dfrac{y}{200}$, or equivalently $y' + \frac{1}{200}\,y = \frac{3}{4}$. This DE is linear (its also separable). An integrating factor is $\lambda = e^{\int \frac{1}{200}\,dt} = e^{t/200}$, and the general solution to the DE is

$$y(t) = \frac{1}{\lambda(t)}\int \tfrac{3}{4}\lambda(t)\,dt = e^{-t/200}\int \tfrac{3}{4}e^{t/200}\,dt = e^{-t/200}\big(150\,e^{t/200} + c\big) = 150 + c\,e^{-t/200}\,.$$

Since $y(0) = 20$ we have $150 + c = 20$ and so $c = -130$. Thus the solution is

$$y(t) = 150 - 130\,e^{-t/200}\,.$$

After 5 hours (that's 300 minutes), we have $y(300) = 150 - 130\,e^{-3/2} \cong 121$, so there will be about 121 kg of salt in the tank.

5.31 Note: According to **Toricelli's Law**, when a liquid drains through a hole in a tank of liquid, it flows through the hole at a speed which is proportional to the square root of the depth of the water above the hole. If the liquid is non-viscous, the speed is

$$v \cong \sqrt{2gy}$$

where g is the gravitational constant and y is the depth.

5.32 Example: A tank, in the shape of an inverted cone of radius 1 m and height 4 m, is filled with water. Water then drains from a hole of area 25 cm^2 at the bottom tip of the tank. If the water drains at a velocity of $v = 4\sqrt{y}$ m/s, where y m is the depth of the water in the tank, then find the time at which the tank will be empty.

Solution: Since the water drains at speed $v = 4\sqrt{y}$ from a hole of area $A = \frac{25}{10000} = \frac{1}{400}$ (in m^2), we have $V' = -Av = -\frac{1}{100}\sqrt{y}$, where $V = V(t)$ is the volume of water in the tank at time t. On the other hand, when the water is y m deep, the water in the tank forms a cone of height y and radius $\frac{1}{4}\,y$, so the volume is $V = \frac{1}{3}\,\pi\left(\frac{1}{4}\,y\right)^2 y = \frac{1}{48}\,\pi y^3$, and so we have $V' = \frac{1}{16}\,\pi y^2\,y'$. Equating these two expressions for V' we find that $\frac{1}{16}\,\pi y^2\,y' = -\frac{1}{100}\,y^{1/2}$, so y satisfies the DE $\frac{\pi}{16}\,y^2\,y' = -\frac{1}{100}\sqrt{y}$ which we can write as $y^{\frac{3}{2}}\,dy = -\frac{4}{25\pi}\,dt$. Integrate both sides to get $\frac{2}{5}\,y^{5/2} = -\frac{4}{25\pi}\,t + c$. Put in $y(0) = 4$ to get $c = \frac{2}{5}\cdot 32$, so we have $\frac{2}{5}\,y^{5/2} = \frac{2}{5}\cdot 32 - \frac{4}{25\pi}\,t$, that is $y = \big(32 - \frac{2}{5\pi}\,t\big)^{2/5}$. The tank will be empty when $y = 0$, and this happens when $\frac{2}{5\pi}\,t = 32$, that is when $t = 80\pi$ so it takes 80π seconds (that is about 4 minutes and 11 seconds) for the tank to empty.