Chapter 6. Parametric and Polar Equations

Parametric Curves

6.1 Definition: For a function $f : I \subseteq \mathbb{R} \to \mathbb{R}^2$, where I is an interval in $\mathbb{R}$, we define the **graph** of f to be the set

$$\text{Graph}(f) = \left\{ \big(x, f(x)\big) \big| x \in I \right\}.$$

When f is continuous, this set is a **curve** in $\mathbb{R}^2$, and we call it the curve defined **explicitly** by the equation $y = f(x)$, or simply the curve $y = f(x)$.

For a function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, where U is a connected set in $\mathbb{R}^2$, we define the **null set** of f to be the set

$$\text{Null(f)} = \left\{ (x, y) \in U \big| f(x, y) = 0 \right\}.$$

When f is continuous, this set is typically a curve in $\mathbb{R}^2$, and we call it the curve defined **implicitly** by the equation $f(x, y) = 0$, or simply the curve $f(x, y) = 0$.

For a function $f : I \subset \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = \big(x(t), y(t)\big)$, where I is an interval in $\mathbb{R}$, we define the **range** (or **image**) of f to be the set

$$\text{Range}(f) = \left\{ f(t) \big| t \in I \right\} = \left\{ \big(x(t), y(t)\big) \big| t \in I \right\}.$$

When $x(t)$ and $y(t)$ are continuous, this set is typically a curve, and we call it the curve defined **parametrically** by the equation $p = (x, y) = f(t)$ $\big($or by the equations $x = x(t)$ and $y = y(t)\big)$, or simply the curve $p = f(t)$. The variable t is called the **parameter**.

6.2 Example: The circle of radius 1 centred at $(0, 0)$ is the curve defined implicitly by the equation

$$x^2 + y^2 = 1.$$

The top half of the circle is given explicitly by the equation

$$y = \sqrt{1 - x^2}\ ,\ -r \leq x \leq r.$$

The entire circle can also be described parametrically by the equation

$$(x, y) = \big(\cos t, \sin t\big)\ ,\ 0 \leq t \leq 2\pi.$$

6.3 Remark: These concepts can be generalized to higher dimensions. For example, the top half of the sphere of radius 1 centred at $(0, 0, 0)$ can be given explicitly by the equation $z = \sqrt{1 - x^2 - y^2}$, and the entire sphere can be given implicitly by $x^2 + y^2 + z^2 = 1$, and the entire sphere can be given parametrically by expressing the coordinates x, y and z of a point on the sphere in terms of the point's latitude ϕ and longitude θ; in mathematics it is common practice to take $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$ with $\phi = 0$ at the north pole and $\phi = \pi$ at the south pole, and then x, y and z are given by

$$(x, y, z) = \big(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi\big).$$

6.4 Example: Given two points $a, b \in \mathbb{R}^2$, the line segment from a to b is given parametrically by

$$p = f(t) = a + t(b - a)\ ,\ 0 \leq t \leq 1.$$

6.5 Example: An arc of the circle of radius r centred at (a, b) can be given parametrically by

$$(x, y) = \big(a + r\cos t, b + r\sin t\big) \text{ with } \alpha \le t \le \beta.$$

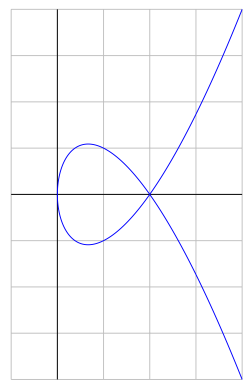
Taking $\alpha = 0$ and $\beta = 2\pi$ yields the entire circle.

6.6 Note: We can always sketch a parametric curve simply by making a table of values and plotting points.

6.7 Example: Sketch the the parametric curve $(x, y) = \big(t^2, t^3 - 2t\big)$.

Solution: We make a table of values and plot points. The points should be connected in order according to the value of t.

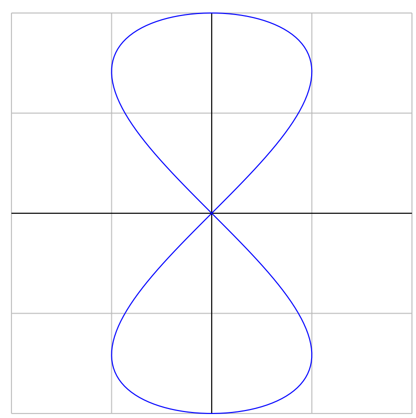
t	x	y
-2	4	-4
$-\sqrt{2}$	2	0
-1	1	1
0	0	0
1	1	-1
$\sqrt{2}$	2	0
2	4	4



6.8 Example: Sketch the parametric curve $(x, y) = \big(\sin 2t, 2\sin t\big)$.

Solution: We make stable of values and plot points.

t	x	y
0	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	1
$\frac{\pi}{4}$	1	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	0	2
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{3\pi}{4}$	-1	$\sqrt{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	1
π	0	0



6.9 Definition: The curve $(x, y) = \big(t^2, t^3 - t\big)$ is called an **alpha curve** because it is shaped like the greek letter α. The curve $(x, y) = \big(\sin 2t, 2\sin t\big)$ is called a **figure eight curve**.

6.10 Note: Sometimes (but not always), given a parametric equation for a curve, we can eliminate the parameter (using some algebraic manipulations) to obtain an implicit or an explicit equation for the curve.

6.11 Example: Eliminate the parameter to find an implicit equation for each of the curves $(x, y) = (t^2 + 1, t^3)$ and $(x, y) = (\sin t, \sec t)$.

Solution: For the curve $(x, y) = (t^2 + 1, t^3)$, we have $(x - 1)^3 = (t^2)^3 = t^6 = y^2$, so the curve is given implicitly by $(x - 1)^3 = y^2$. For the curve $(x, y) = (\sin t, \sec t)$ we have $y^2 = \sec^2 t = \frac{1}{\cos^2 t} = \frac{1}{1-\sin^2 t} = \frac{1}{1-x^2}$ and so the curve is given implicitly by $y^2 = \frac{1}{1-x^2}$.

6.12 Example: Eliminate the parameter to find an implicit equation for the alpha curve $(x, y) = \big(t^2, t^3 - 2t\big)$ and for the figure eight curve $(x, y) = (\sin 2t, 2 \sin t)$.

Solution: For the alpha curve $(x, t) = \big(t^2, t^3 - 2t\big)$ we have

$$y^2 = (t^3 - 2t)^2 = (t^6 - 4t^4 + 4t^2 = x^3 - 4x^2 + 4x = x(x-2)^2$$

and so this alpha curve is given implicitly by $y^2 = x(x-2)^2$. For the figure eight curve $(x, y) = \big(\sin 2t, 2\sin t\big)$ we have

$$\begin{aligned} x^2 &= (\sin 2t)^2 = (2 \sin t \cos t)^2 = 4 \sin^2 t \cos^2 t = 4 \sin^2 t (1 - \sin^2 t) \\ &= y^2\big(1 - (y/2)^2\big) = \tfrac{1}{4}\, y^2(4 - y^2) = \tfrac{1}{4}\, y^2(2-y)(2+y)\,, \end{aligned}$$

and so this figure eight curve is given implicitly by $x^2 = \frac{1}{4}\, y^2(2-y)(2+y)$.

6.13 Note: For a parametric curve $(x, y) = f(t) = \big(x(t), y(t)\big)$, when the variable t represents time, the point $f(t) = \big(x(t), y(t)\big)$ represents the position of a moving point.

6.14 Example: A **cycloid** is the curve followed by a point on a circle in the xy-plane which rolls, without slipping, along the x-axis. Find a parametric equation for a cycloid

Solution: Say the circle has radius r, it begins with its centre at position $(0, r)$, and it rolls in the direction of the positive x-axis at speed s, and say we are interested with the point on the circle initially at position $(0, 0)$. At time t, the centre will be at position (st, r). Let $\theta = \theta(t)$ be the angle through which the circle has revolved about its centre at time t. Since the circle revolves at a constant rate, we have $\theta(t) = ct$ for some constant c. Since the circle rolls without slipping, it makes one full revolution about its centre when $x(t) = 2\pi r$ (the circumference of the circle), so we have $\theta(t) = 2\pi$ when $st = 2\pi r$, that is $ct = 2\pi$ when $t = 2\pi\, r/s$, and so $c = s/r$. Since, at time t, the centre of the circle is at $(st, 0) = (r\theta(t), r)$ and the circle has rotated (clockwise) by the angle $\theta(t) = \frac{s}{r}\, t$, it follows that the point on the circle which was originally at $(0, 0)$ will have moved to the position

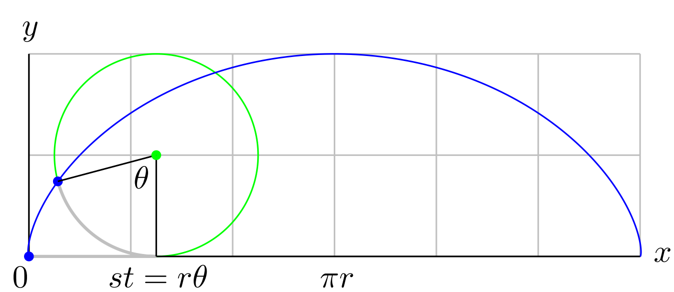
$$(x, y) = (r\,\theta(t), r) - \big(r \sin \theta(t), r \cos \theta(t)\big)\,.$$

We can use the angle θ as our parameter and write this as

$$(x, y) = \big(x(\theta), y(\theta)\big) = (r\theta, r) - (r \sin \theta, r \cos \theta) = r\big(\theta - \sin \theta, 1 - \cos \theta\big)$$

or we can use time t as our parameter and write

$$(x, y) = \big(x(t), y(t)\big) = (st, r) - \Big(r \sin\left(\tfrac{s}{r}\, t\right), r \cos\left(\tfrac{s}{r}\, t\right)\Big)$$



6.15 Definition: The **tangent vector** to the parametric curve $(x,y)=f(t)=\big(x(t),y(t)\big)$ at the point where $t=t_0$ is the vector

$$f'(t_0)=\big(x'(t_0),y'(t_0)\big)\,.$$

The **linearization** of f at $t=t_0$ is the function $L(t)$ given by

$$L(t)=f(t_0)+f'(t_0)(t-t_0)$$

and when $f'(t_0)\neq 0$, the **tangent line** to the curve $p=f(t)$ at the point $f(t_0)$ is the line given parametrically by

$$(x,y)=L(t)=f(t_0)+f'(t_0)(t-t_0)\,.$$

When t represents time and $f(t)$ represents the position of a moving point, the tangent vector $f'(t)=\big(x'(t),y'(t)\big)$ is also called the **velocity** of the moving point at time t. The **speed** of the moving point is the length of the velocity vector. We also define the **acceleration** of the moving point at time t to be the vector $f''(t)=\big(x''(t),y''(t)\big)$.

6.16 Example: Consider the alpha curve $(x,y)=\big(t^2,t^3-2t\big)$. Find an explicit equation for the tangent line at the point where $t=1$.

Solution: We have $\big(x(t),y(t)\big)=(t^2,t^3-2t)$ and so $\big(x'(t),y'(t)\big)=(2t,3t^2-2)$. When $t=1$ we have $(x,y)=(1,-1)$ and $(x',y')=(2,1)$, and so the tangent line is the line through $(1,-1)$ in the direction of the vector $(2,1)$. This line has slope $\frac{1}{2}$, and its equation is $y+1=\frac{1}{2}(x-1)$, that is $y=\frac{1}{2}\,x-\frac{3}{2}$.

6.17 Example: A small stone is stuck in the tread of the tire of a car. The tire has radius $r=0.25$ (in meters) and the car moves at speed $s=10$ (in meters per second). The stone moves along a cycloid with its position (in meters) at time t (in seconds) given by

$$(x,y)=\big(x(t),y(t)\big)=(st,r)-\left(r\sin\left(\tfrac{s}{r}\,t\right),r\cos\left(\tfrac{s}{r}\,t\right)\right).$$

Find the position, the velocity, and the speed of the stone at time $t=\pi/120$.

Solution: Put $r=\frac{1}{4}$ and $s=10$ into the parametric equations to get

$$(x,y)=\big(10\,t,\tfrac{1}{4}\big)-\big(\tfrac{1}{4}\sin 40t,\tfrac{1}{4}\cos 40t\big)$$
$$(x',y')=\big(10,0\big)-\big(10\cos 40t,-10\sin 40t\big)\,.$$

When $t=\frac{\pi}{120}$ the position, velocity and speed are

$$p=(x,y)=\big(\tfrac{\pi}{12},\tfrac{1}{4}\big)-\big(\tfrac{\sqrt{3}}{8},\tfrac{1}{8}\big)=\big(\tfrac{\pi}{12}-\tfrac{\sqrt{3}}{8},\tfrac{1}{8}\big)$$
$$v=(x',y')=\big(10,0\big)-\big(5,-5\sqrt{3}\big)=(5,5\sqrt{3})$$
$$|v|=\sqrt{(x')^2+(y')^2}=\sqrt{25+75}=10\,.$$

Is it surprising that the stone is moving at the same speed as the car?

6.18 Note: Consider the parametric curve $(x,y)=f(t)=\big(x(t),y(t)\big)$ with $r\le t\le s$. Suppose that we are able to eliminate the parameter to express the curve explicitly by $y=g(x)$. Then for all $t\in[r,s]$ we have

$$y(t)=g\big(x(t)\big)\,.$$

Taking the derivative (with respect to t) on both sides, we obtain $y'(t)=g'\big(x(t)\big)x'(t)$ and so

$$g'\big(x(t)\big)=\frac{y'(t)}{x'(t)}$$

whenever $x'(t)\neq 0$. This formula should come as no surprise because both sides measure the slope of the tangent line to the given curve at the point $\big(x(t),y(t)\big)$. Taking the derivative again, we obtain $g''\big(x(t)\big)x'(t)=\frac{d}{dt}\big(y'(t)/x'(t)\big)$, that is

$$g''\big(x(t)\big)=\frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}$$

whenever $x'(t)\neq 0$. We could also obtain a formula for $g'''\big(x(t)\big)$ in terms of $x(t)$ and $y(t)$.

6.19 Example: Consider the figure-eight curve $(x,y)=\big(\sin 2t, 2\sin t\big)$. Suppose that the portion of the curve with $-\frac{\pi}{4}\le t\le\frac{\pi}{4}$ is given explicitly by $y=g(x)$ with $-1\le x\le 1$. Find $g'\big(\frac{\sqrt3}{2}\big)$ and $g''\left(\frac{\sqrt3}{2}\right)$.

Solution: Note that for $-\frac{\pi}{4}\le t\le\frac{\pi}{4}$ we have

$$x(t)=\tfrac{\sqrt3}{2}\iff \sin 2t=\tfrac{\sqrt3}{2}\iff 2t=\tfrac{\pi}{3}\iff t=\tfrac{\pi}{6}\,.$$

Also, we have

$$\begin{aligned}
\big(x(t),y(t)\big)&=\big(\sin 2t, 2\sin t\big)\\
\big(x'(t),y'(t)\big)&=\big(2\cos 2t, 2\cos t\big)\\
g'\big(x(t)\big)&=\frac{y'(t)}{x'(t)}=\frac{\cos t}{\cos 2t}\\
g''\big(x(t)\big)&=\frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}=\frac{\frac{d}{dt}\left(\frac{\cos t}{\cos 2t}\right)}{2\cos 2t}=\frac{-\sin t\cos 2t+2\cos t\sin 2t}{2\cos^3 2t}\,.
\end{aligned}$$

Put in $t=\frac{\pi}{6}$ to get

$$\begin{aligned}
g'\left(\tfrac{\sqrt3}{2}\right)&=\frac{\cos\frac{\pi}{6}}{\cos\frac{\pi}{3}}=\frac{\frac{\sqrt3}{2}}{\frac12}=\sqrt3\\
g''\left(\tfrac{\sqrt3}{2}\right)&=\frac{-\sin\frac{\pi}{6}\cos\frac{\pi}{3}+2\cos\frac{\pi}{6}\sin\frac{\pi}{3}}{2\cos^3\frac{\pi}{3}}=\frac{-\frac12\cdot\frac12+2\cdot\frac{\sqrt3}{2}\cdot\frac{\sqrt3}{2}}{\frac14}=5\,.
\end{aligned}$$

6.20 Note: Consider the curve is given parametrically by $(x,y)=f(t)=\big(x(t),y(t)\big)$ with $r\leq t\leq s$. Suppose that $y(t)\geq 0$ and $x'(t)\geq 0$ for all $t\in[r,s]$ and let $a=x(r)$ and $b=x(s)$. Note that $a\geq b$ since $x'(t)\geq 0$ for all t. Suppose that we can eliminate the parameter to express the curve explicitly by $y=g(x)$ with $a\leq x\leq b$. Note that for all $t\in[r,s]$ we have $y(t)=g\big(x(t)\big)$. Using the Substitution Rule, we obtain the following formulas.

The area of the region R given by $a\leq x\leq b$, $0\leq y\leq g(x)$ is

$$A=\int_{x=a}^{b} g(x)\,dx=\int_{t=r}^{s} g\big(x(t)\big)\,x'(t)\,dt=\int_{t=r}^{s} y(t)\,x'(t)\,dt\,,$$

the volume of the solid obtained by revolving R about the x-axis is

$$V=\int_{x=a}^{b}\pi\,g(x)^2\,dx=\int_{t=r}^{s}\pi\,y(t)^2x'(t)\,dt\,,$$

in the case that $a\geq 0$, the volume of the solid obtained by revolving R about the y-axis is

$$V=\int_{x=a}^{b}2\pi\,xg(x)\,dx=\int_{t=r}^{s}2\pi\,x(t)\,y(t)\,x'(t)\,dt\,,$$

the length of the curve C given by $y=g(x)$ with $a\leq x\leq b$ is

$$L=\int_{x=a}^{b}\sqrt{1+g'(x)^2}\;dx=\int_{t=r}^{s}\sqrt{1+\tfrac{y'(t)^2}{x'(t)^2}}\;x'(t)\,dt=\int_{t=r}^{s}\sqrt{x'(t)^2+y'(t)^2}\;dt\,,$$

the area of the surface obtained by revolving C about the x-axis is

$$A=\int_{x=a}^{b}2\pi\,g(x)\sqrt{1+g'(x)^2}\;dx=\int_{t=r}^{s}2\pi\,y(x)\sqrt{x'(t)^2+y'(t)^2}\;dt\,,$$

and when $a\geq 0$, the area of the surface obtained by revolving C about the y-axis is

$$A=\int_{x=a}^{b}2\pi\,x\sqrt{1+g'(x)^2}\;dx=\int_{t=r}^{s}2\pi\,x(t)\sqrt{x'(t)^2+y'(t)^2}\;dt\,.$$

6.21 Example: Consider the curve C given by $(x,y)=\big(t^2,t^3\big)$ with $0\leq t\leq 2$. Find the area of the region R which lies below C and above the x-axis between $x=0$ and $x=4$. Find the volume of the solid obtained by revolving R about the x-axis. Find the length of the curve C.

Solution: The required area, volume and length are

$$A=\int_0^2 y(t)\,x'(t)\,dt=\int_0^2 t^3\cdot 2t\;dt=\int_0^2 2t^4\;dt=\Big[\tfrac{2}{5}\,t^5\Big]_0^2=\tfrac{64}{5}$$

$$V=\int_{t=0}^{2}\pi\,y(t)^2\,x'(t)\,dt=\int_0^2\pi\,t^6\cdot 2t\;dt=\int_0^2 2\pi\,t^7\;dt=\Big[\tfrac{\pi}{4}\,t^8\Big]_0^2=64\pi$$

$$L=\int_{t=0}^{2}\sqrt{x'(t)^2+y'(t)^2}\;dt=\int_0^2\sqrt{(2t)^2+(3t^2)^2}\;dt=\int_0^2 t\sqrt{4+9t^2}\;dt\,.$$

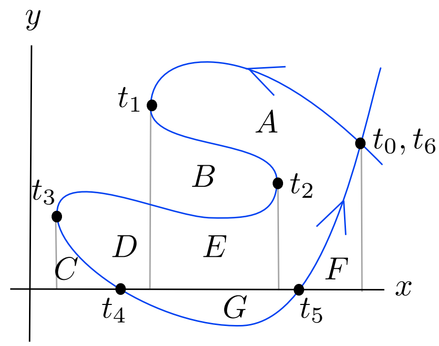
To solve the last integral, let $u=4+9t^2$ so that $du=18t\,dt$. Then the length is

$$L=\int_{t=0}^{2}t\sqrt{4+9t^2}\;dt=\int_{u=4}^{40}\tfrac{1}{18}\,u^{1/2}\,du=\Big[\tfrac{1}{27}\,u^{3/2}\Big]_4^{40}=\tfrac{1}{27}\big(40^{3/2}-8\big)\,.$$

6.22 Note: We can obtain similar formulas to the ones above in the case that $y(t) \leq 0$ or in the case that $x'(t) \leq 0$. For example, when $y(t) \geq 0$ and $x'(t) \leq 0$ for all $t \in [r, s]$, if we let $a = x(s)$ and $b = x(r)$ so that $a \leq b$, then the area of the region R given by $a \leq x \leq b$, $0 \leq y \leq g(x)$ is equal to

$$A = \int_{x=a}^{b} g(x)\, dx = \int_{t=s}^{r} y(t)\, x'(t)\, dt = -\int_{t=r}^{s} y(t)\, x'(t)\, dt\,.$$

6.23 Example: Consider the parametric curve $(x, y) = \big(x(t), y(t)\big)$ with $r \leq t \leq s$ whose image is shown below. Let $r = t_0 < t_1 < \cdots < t_6 = s$ be the values of t corresponding to the indicated points, and let A, B, C, D, E, F and G be the areas of the indicated regions.



For $t_0 \leq t \leq t_1$ we have $y(t) \geq 0$ and $x'(t) \leq 0$ so the integral $\displaystyle\int_{t_0}^{t_1} y(y)\, x'(t)\, dt$ measures the *negative* of the area under the curve, that is $\displaystyle\int_{t_0}^{t_1} y(y)\, x'(t)\, dt = -(A + B + E + F)\,.$ For $t_1 \leq t \leq t_2$ we have $y(t) \geq 0$ and $x'(t) \geq 0$ so the integral $\displaystyle\int_{t_1}^{t_2} y(y)\, x'(t)\, dt$ measures the positive area under the curve, that is $\displaystyle\int_{t_1}^{t_2} y(y)\, x'(t)\, dt = (B + E)\,.$ Similarly, we find that $\displaystyle\int_{t_2}^{t_3} y(t)\, x'(t)\, dt = -(C + D + E)$, $\displaystyle\int_{t_3}^{t_4} y(t)\, x'(t)\, dt = C$, $\displaystyle\int_{t_4}^{t_5} y(t)\, x'(t)\, dt = -G$, and $\displaystyle\int_{t_5}^{t_6} y(t)\, x'(t)\, dt = F$. Thus we have

$$\begin{aligned}\int_r^s y(t)\, x'(t)\, dt &= -(A + B + E + F) + (B + E) - (C + D + E) + (C) - (G) + (F) \\ &= -(A + D + E + G)\end{aligned}$$

so the integral measures the negative of the area inside the loop. If the loop had been traversed in the opposite direction (clockwise instead of anticlockwise) the integral would have given the positive area inside the loop.

6.24 Example: Consider the alpha curve given by $(x,y) = \big(t^2, t^3 - 2t\big)$. Find the area of the region R inside the loop, and find the volume of the solid obtained by revolving R about the x-axis.

Solution: With the help of the table of values and plot from example 6.7, we see that the loop is the portion of the curve with $-\sqrt{2} \le t \le \sqrt{2}$. By symmetry, the bottom half of the region R lies between the axis and the portion of the curve with $0 \le t \le \sqrt{2}$. For that part of the curve we have $y(t) = t(t-\sqrt{2})(t+\sqrt{2}) \le 0$ and $x'(t) = 2t \ge 0$ and so the area of R is

$$\begin{aligned} A &= -2\int_{t=0}^{\sqrt{2}} y(t)\, x'(t)\, dt = -2\int_0^{\sqrt{2}} (t^3 - 2t)\cdot 2t\, dt = 4\int_0^{\sqrt{2}} 2t^2 - t^4\ dt \\ &= 4\Big[\tfrac{2}{3}\,t^3 - \tfrac{1}{5}\,t^5\Big]_0^{\sqrt{2}} = 4\big(\tfrac{4\sqrt{2}}{3} - \tfrac{4\sqrt{2}}{5}\big) = \tfrac{32\sqrt{2}}{15} \end{aligned}$$

and (since $x'(t) > 0$) the volume of the solid obtained by revolving R about the x-axis is

$$\begin{aligned} V &= \int_{t=0}^{\sqrt{2}} \pi\, y(t)^2 x'(t)\, dt = \int_0^{\sqrt{2}} \pi(t^3 - 2t)^2\cdot 2t\, dt = 2\pi\int_0^{\sqrt{2}} t^7 - 4t^5 + 4t^3\ dt \\ &= 2\pi\Big[\tfrac{1}{8}\,t^8 - \tfrac{2}{3}\,t^6 + t^4\Big]_0^{\sqrt{2}} = 2\pi\big(2 - \tfrac{16}{3} + 4\big) = \tfrac{4\pi}{3}\,. \end{aligned}$$

6.25 Example: Consider the cycloid $(x,y) = (t - \sin t, 1 - \cos t)$. Find the length of one arch of the cycloid, and find the area of the surface obtained by revolving this arch about the x-axis.

Solution: Note that one arch of the cycloid is the part of the cycloid given by $0 \le t \le 2\pi$. We have $x'(t) = 1 - \cos t$ and $y'(t) = \sin t$ and so for $0 \le t \le 2\pi$ we have

$$\begin{aligned} \sqrt{x'(t)^2 + y'(t)^2} &= \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2 - 2\cos t} = \sqrt{4\sin^2\left(\tfrac{1}{2}\,t\right)} = 2\sin\left(\tfrac{1}{2}\,t\right) \end{aligned}$$

(since $\sin(t/2) \ge 0$ for $0 \le t \le 2\pi$) and so the length of one arch is

$$L = \int_{t=0}^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\ dt = \int_0^{2\pi} 2\sin\left(\tfrac{1}{2}\,t\right)\ dt = \Big[-4\cos\left(\tfrac{1}{2}\,t\right)\Big]_0^{2\pi} = 4 + 4 = 8$$

and the area of the surface obtained by revolving this arch about the x-axis is

$$\begin{aligned} A &= \int_{t=0}^{2\pi} 2\pi\, y(t)\sqrt{x'(t)^2 + y'(t)^2}\ dt = \int_0^{2\pi} 2\pi(1 - \cos t)\cdot 2\sin\left(\tfrac{1}{2}\,t\right)\ dt \\ &= 4\pi\int_0^{2\pi} \Big(1 - \big(1 - 2\sin^2\left(\tfrac{1}{2}\,t\right)\big)\Big)\sin\left(\tfrac{1}{2}\,t\right)\ dt = 8\pi\int_0^{2\pi} \sin^3\left(\tfrac{1}{2}\,t\right)\ dt\,. \end{aligned}$$

To solve the integral, we let $u = \cos\left(\tfrac{1}{2}\,t\right)$ so that $du = -\tfrac{1}{2}\sin\left(\tfrac{1}{2}\,t\right)\ dt$.Then

$$\begin{aligned} A &= 8\pi\int_{t=0}^{2\pi} \sin^3\left(\tfrac{1}{2}\,t\right)\ dt = 8\pi\int_{t=0}^{2\pi} \Big(1 - \cos^2\left(\tfrac{1}{2}\,t\right)\Big)\sin\left(\tfrac{1}{2}\,t\right)\ dt \\ &= 8\pi\int_{u=1}^{-1} -2(1 - u^2)\, du = -16\pi\Big[u - \tfrac{1}{3}\,u^3\Big]_1^{-1} = -16\pi\big(-\tfrac{2}{3} - \tfrac{2}{3}\big) = \tfrac{64\pi}{3}\,. \end{aligned}$$

Polar Coordinates

6.26 Definition: A point in the plane is most commonly represented by an ordered pair (x,y) with $x\in\mathbb{R}$ and $y\in\mathbb{R}$, where x represents the horizontal position of the point and y represents the vertical position. The numbers x and y are called the **cartesian coordinates** of the point. To plot the position of a point represented in cartesian coordinates, it is convenient to use a **cartesian grid** which usually shows the x-axis pointing to the right and the y-axis pointing upwards along with some horizontal lines $y=$constant and some vertical lines $x=$constant.

The second most common way to represent a point in the plane is by an ordered pair (r,θ) with $0\leq r\in\mathbb{R}$ and $\theta\in\mathbb{R}$ where r represents the distance from the point to the origin and (when $r\neq 0$) θ represents the angle from the positive x-axis to the point in the counterclockwise direction. The numbers r and θ are called the **polar coordinates** of the point. To plot the position of a point represented i polar coordinates, it is convenient to use a **polar grid** which usually shows the x and y-axes along with some circles $r=$constant and some rays $\theta=$constant.

6.27 Note: When a point is represented in cartesian coordinates by (x,y) and in polar coordinates by (r,θ), the coordinates x, y, r and θ satisfy the following relationships.

$$\begin{array}{ll} x=r\cos\theta & r^2=x^2+y^2 \\ y=r\sin\theta & \tan\theta=\frac{y}{x} \end{array}$$

6.28 Note: Given a point in the plane, the values of its cartesian coordinates x and y uniquely determine and are uniquely determined by the position of the point. On the other hand, although the values of the polar coordinates r and θ uniquely determine the position of the point and the position of the point uniquely determines the value of r, the position of the point does not uniquely determine the value of θ. At the origin $(x,y)=(0,0)$, we have $r=0$ while $\theta\in\mathbb{R}$ is arbitrary, and at points $(x,y)\neq(0,0)$, we have $r=\sqrt{x^2+y^2}$ and θ is only determined uniquely up to a multiple of 2π. To be explicit, we have

$$\theta=\begin{cases} \tan^{-1}\frac{y}{x} & +2\pi\,k \quad \text{for some } k\in\mathbb{Z} & \text{, if } x>0 \\ \cos^{-1}\frac{x}{\sqrt{x^2+y^2}} & +2\pi\,k \quad \text{for some } k\in\mathbb{Z} & \text{, if } y>0 \\ \pi+\tan^{-1}\frac{y}{x} & +2\pi\,k \quad \text{for some } k\in\mathbb{Z} & \text{, if } x>0 \\ -\cos^{-1}\frac{x}{\sqrt{x^2+y^2}} & +2\pi\,k \quad \text{for some } k\in\mathbb{Z} & \text{, if } y>0 \end{cases}$$

6.29 Note: Sometimes, when expressing points in polar coordinates, we allow r to take negative values. For all values of $r\in\mathbb{R}$, the point given in polar coordinates by (r,θ) corresponds to the point given in cartesian coordinates by $(x,y)=\big(r\cos\theta,r\sin\theta\big)$.

6.30 Example: Express each of the polar points $(r,\theta)=\big(2,-\frac{\pi}{6}\big)$, $\big(-6,\frac{3\pi}{4}\big)$ and $\big(10,\tan^{-1}3\big)$ in cartesian form.

Solution: The point given in polar coordinates by $(r,\theta)=\big(2,-\frac{\pi}{6}\big)$ is given in cartesian coordinates by $(x,y)=(\sqrt{3},-1)$. The polar point given by $(r,\theta)=\big(-6,\frac{3\pi}{4}\big)$ is given in cartesian coordinates by $(x,y)=(3\sqrt{2},-3\sqrt{2})$. Note that when $\theta=\tan^{-1}3$ we have $\sin\theta=\frac{3}{\sqrt{10}}$ and $\cos\theta=\frac{1}{\sqrt{10}}$, and so the polar point $(r,\theta)=(10,\tan^{-1}3)$ is given in cartesian coordinates by $(x,y)=\big(\sqrt{10},3\sqrt{10}\big)$.

6.31 Example: Express each of the cartesian points $(x,y)=(1,\sqrt{3})$, $(-2,2)$ and $(-3,-4)$ in polar form.

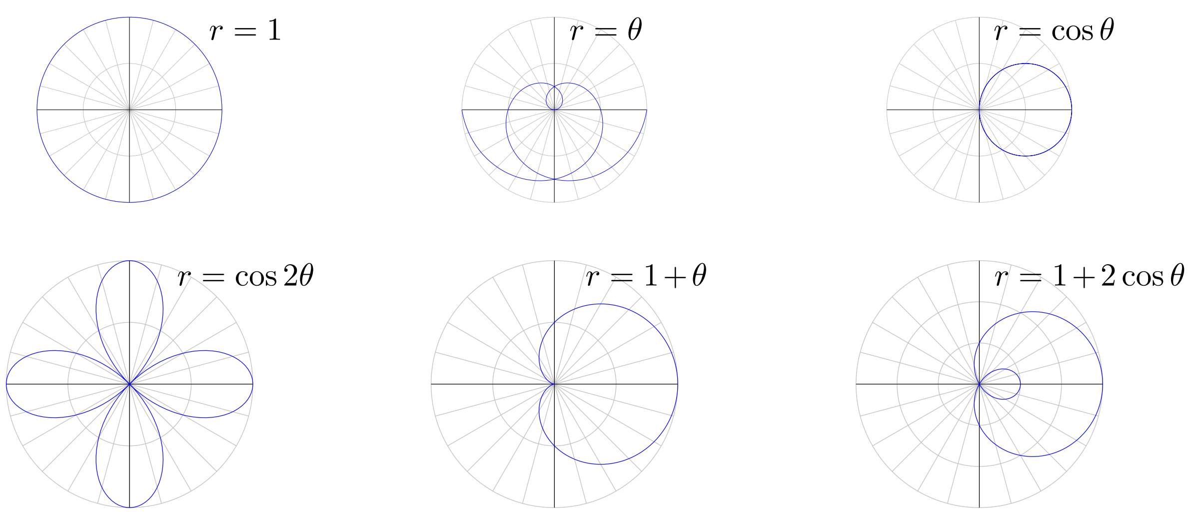
Solution: The point given in cartesian coordinates by $(x,y)=(1,\sqrt{3})$ can be given in polar coordinates by $(r,\theta)=\left(2,\frac{\pi}{3}\right)$. The cartesian point $(x,y)=(-2,2)$ can be given in polar coordinates by $(r,\theta)=\left(2\sqrt{2},\frac{3\pi}{4}\right)$. The cartesian point $(x,y)=(-3,-4)$ can be given in polar coordinates by $(r,\theta)=\left(5,\pi+\tan^{-1}\frac{4}{3}\right)$. The third point $(x,y)=(-3,-4)$ could also be given, for example, by $(r,\theta)=\left(-5,\tan^{-1}\frac{4}{3}\right)=\left(-5,\sin^{-1}\frac{4}{5}\right)$.

6.32 Note: A curve in the plane can be described in polar coordinates either explicitly, implicitly or parametrically. The explicit curve given by $r=f(\theta)$ for $\theta\in I$ is the set of polar points $\left\{(r,\theta)\big|r=f(\theta)\text{ for some }\theta\in I\right\}$, the implicit curve $f(r,\theta)=0$ is the set of polar points $\left\{(r,\theta)\big|f(r,\theta)=0\right\}$, and the parametric curve given by $(r,\theta)=f(t)=\big(r(t),\theta(t)\big)$ for $t\in I$ is the set of polar points $\left\{\big(r(t),\theta(t)\big)\big|t\in I\right\}$. Such curves are most easily sketched using a polar grid.

6.33 Example: Sketch each of the explicit polar curves $r=1$, $r=\theta$, $r=\cos\theta$, $r=\cos 2\theta$, $r=1+\cos\theta$ and $r=1+2\cos\theta$.

Solution: For each curve, we make a table of values and plot points in a polar grid.

θ	$r=1$	$r=\theta$	$r=\cos\theta$	$r=\cos 2\theta$	$r=1+\cos\theta$	$r=1+2\cos\theta$
0	1	0	1	1	2	3
$\pi/6$	1	$\pi/6$	$\sqrt{3}/2$	$1/2$	$1+\sqrt{3}/2$	$1+\sqrt{3}$
$\pi/4$	1	$\pi/4$	$\sqrt{2}/2$	0	$1+\sqrt{2}/2$	$1+\sqrt{2}$
$\pi/3$	1	$\pi/3$	$1/2$	$-1/2$	$3/2$	2
$\pi/2$	1	$\pi/2$	0	-1	1	1
$2\pi/3$	1	$2\pi/3$	$-1/2$	$-1/2$	$1/2$	0
$3\pi/4$	1	$3\pi/4$	$-\sqrt{2}/2$	0	$1-\sqrt{2}/2$	$1-\sqrt{2}$
$5\pi/6$	1	$5\pi/6$	$-\sqrt{3}/2$	$1/2$	$1-\sqrt{3}/2$	$1-\sqrt{3}$
π	1	π	-1	1	0	-2



6.34 Definition: A **limaçon** is a polar curve of the form $r=a+b\cos\theta$ (or $r=a+b\sin\theta$), where $a,b\in\mathbb{R}^+$. A **cardioid** is a limaçon of the form $r=a+a\cos\theta$ (or $r=a+a\sin\theta$). A polar **rose** is a polar curve of the form $r=a\cos(n\theta)$ $\big($or $r=a\sin(n\theta)\big)$ where $a\in\mathbb{R}^+$ and $n\in\mathbb{Z}^+$. When n is odd, the polar rose $r=a\cos(n\theta)$ has n petals, but when n is even, the polar rose has $2n$ petals,

6.35 Note: If a curve is described explicitly in cartesian coordinates by $y = f(x)$, then the same curve is described implicitly in polar coordinates (using the formulas $x = r\cos\theta$ and $y = r\sin\theta$) by $r\sin\theta = f(r\cos\theta)$.

If a curve is described implicitly in cartesian coordinates by $f(x,y) = 0$, then the same curve is given implicitly in polar coordinates by $f\big(r\cos\theta, r\sin\theta\big) = 0$.

If a curve is described parametrically in cartesian coordinates by $(x,y) = \big(x(t), y(t)\big)$ then sometimes (but not always) we can use algebraic manipulation (making use of the formulas $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$) to express the curve in polar coordinates.

6.36 Note: If a curve is given parametrically in polar coordinates by $(r,\theta) = \big(r(t), \theta(t)\big)$, then it is given parametrically in cartesian coordinates by $(x,y) = \big(r(t)\cos\theta(t), r(t)\sin\theta(t)\big)$.

If a curve is described explicitly in polar coordinates by $r = r(\theta)$ then the same curve is given parametrically, still in polar coordinates, by $(r,\theta) = \big(r(t), t\big)$, and so it is given parametrically in cartesian coordinates by $(x,y) = \big(r(t)\cos t, r(t)\sin t\big)$.

If a curve is given implicitly in polar coordinates by $f(r,\theta) = 0$, then sometimes (but not always) we can use algebraic manipulation to express the curve in cartesian coordinates.

6.37 Example: Express each of the cartesian curves $y = x + 1$, $y = x^2$ and $y^2 - x^2 = 1$ explicitly in polar coordinates.

Solution: The line $y = x + 1$ is given implicitly in polar coordinates by $r\sin\theta = r\cos\theta + 1$, or equivalently by $r(\sin\theta - \cos\theta) = 1$, and so it is given explicitly by

$$r = r(\theta) = \frac{1}{\sin\theta - \cos\theta}\,.$$

The parabola $y = x^2$ is given implicitly in polar coordinates by $r\sin\theta = (r\cos\theta)^2$, that is $r\big(\sin\theta = r\cos^2\theta\big)$. or equivalently $r = 0$ or $r = \sin\theta/\cos^2\theta = \sec\theta\tan\theta$. Since the origin $r = 0$ already lies on the polar curve $r = \sec\theta\tan\theta$, the curve is given explicitly simply by

$$r = r(\theta) = \sec\theta\tan\theta\,.$$

Finally the hyperbola, given in cartesian coordinates by $y^2 - x^2 = 1$, is given implicitly in polar coordinates by $r^2\sin^2\theta - r^2\cos^2\theta = 1$, that is $r^2(\sin^2\theta - \cos^2\theta) = 1$, or equivalently $r^2(-\cos 2\theta) = 1$, that $r^2 = -\sec 2\theta$. When $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, and then again when $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$, we have $\sec 2\theta < 0$ and we can take $r = \sqrt{-\sec 2\theta}$, so the hyperbola is given explicitly by

$$r = r(\theta) = \sqrt{-\sec 2\theta}\,.$$

6.38 Example: Express each of the polar curves $r = 1$, $r = \theta$, $r = \cos\theta$, $r = \cos 2\theta$ and $r = 1 + \cos\theta$ in cartesian coordinates.

Solution: The polar curve $r = 1$ is the unit circle, which can be given in cartesian coordinates, either implicitly by $x^2 + y^2 = 1$, or parametrically by $(x.y) = (\cos t, \sin t)$. The polar curve $r = \theta$ can be given parametrically in polar coordinates by $(r,\theta) = (t,t)$, and it can be given parametrically in cartesian coordinates by

$$(x,y) = (r\cos\theta, r\sin\theta) = (t\cos t, t\sin t)\,.$$

The polar curve $r = \cos\theta$ can also be given in polar coordinates by $r^2 = r\cos\theta$, and in cartesian coordinates this becomes $x^2 + y^2 = x$. Completing the square, the equation can also be written as

$$\big(x - \tfrac{1}{2}\big)^2 + y^2 = \tfrac{1}{4}$$

and so we see that the polar curve $r = \cos\theta$ is in fact the circle of radius $\frac{1}{2}$ centered at $(x,y) = \left(\frac{1}{2}, 0\right)$ (you might have guessed this by looking at its graph, which we plotted in Example 6.33). Finally, the four-petaled rose given in polar coordinates by $r = \cos 2\theta = \cos^2\theta - \sin^2\theta$ can also be given in polar coordinates by $r^3 = (r\cos\theta)^2 + (r\sin\theta)^2$, which we can write in cartesian coordinates as $(x^2+y^2)^{3/2} = x^2 - y^2$, or equivalently as

$$(x^2+y^2)^3 = x^2 - y^2 .$$

6.39 Note: Given a polar curve which is described either explicitly or parametrically, we can describe the curve parametrically in cartesian coordinates, and then we can perform various calculations related to the curve, for example we can find the slope of the curve at a point or we can find the tangent line to the curve at a point, or we can find the area of the region R which lies between the curve and the x-axis with $a \leq x \leq b$, or we can find the volume of the solid obtained by revolving R about either the x or the y-axis.

6.40 Example: Find a formula for the slope of the polar curve $r = r(\theta)$ at the point where $\theta = t$ (that is, the slope of the tangent line to the curve at the point where $\theta = t$).

Solution: The curve can be given parametrically in polar coordinates by $(r,\theta) = \big(r(t), t\big)$ and so it can be given parametrically in cartesian coordinates by

$$(x,y) = \big(r(t)\cos t, r(t)\sin t\big) .$$

Using the Product Rule, the slope at the point where $\theta = t$ is equal to

$$\frac{y'(t)}{x'(t)} = \frac{r'(t)\sin t + r(t)\cos t}{r'(t)\cos t - r(t)\sin t} .$$

6.41 Example: Find the cartesian coordinates of all of the horizontal and vertical points on the cardioid $r = 1 + \cos\theta$.

Solution: The cardioid can be given parametrically in cartesian coordinates by

$$(x,y) = \big((1+\cos t)\cos t, (1+\cos t)\sin t\big) .$$

Since $x(t) = \cos t + \cos^2 t$, we have

$$x'(t) = -\sin t - 2\sin t\cos t = -(\sin t)(1 + 2\cos t)$$

and so $x'(t) = 0$ when $\sin t = 0$ and when $\cos t = -\frac{1}{2}$. that is when $t = 0, \pi, \pm\frac{2\pi}{3}$ plus integer multiples of 2π. Since $y(t) = \sin t + \sin t\cos t$, we have

$$y'(t) = \cos t + \cos^2 t - \sin^2 t = 2\cos^2 t + \cos t - 1 = (2\cos t - 1)(\cos t + 1)$$

and so $y'(t) = 0$ when $\cos t = \frac{1}{2}$ and when $\cos t = -1$, that is when $t = \pm\frac{\pi}{3}, \pi$ plus multiples of 2π. When $t = 0, \pm\frac{2\pi}{3}$, that is at the points $(x,y) = (2,0), \left(-\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)$, we have $x'(t) = 0$ and $y'(t) = 0$, so the curve is vertical at these points. When $t = \pm\frac{\pi}{3}$, that is at the points $(x,y) = \left(\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$, we have $y'(t) = 0$ and $x'(t) \neq 0$, so the curve is horizontal at these points. When $t = \pi$, that is at the point $(x,y) = (0,0)$, we have both $x'(t)$ and $y'(t) = 0$ so some care is needed. Using L'Hôpital's Rule, we have

$$\begin{aligned}\lim_{t\to\pi}\frac{y'(t)}{x'(t)} &= \lim_{t\to\pi}\frac{(2\cos t - 1)(\cos t + 1)}{-(\sin t)(1 + 2\cos t)} = \lim_{t\to\pi}\frac{2\cos t - 1}{1 + 2\cos t}\cdot\lim_{t\to\pi}\frac{\cos t + 1}{-\sin t}\\ &= \tfrac{-2-1}{1-2}\cdot\lim_{t\to\pi}\frac{\cos t + 1}{-\sin t} = 3\cdot\lim_{t\to\pi}\frac{-\sin t}{\cos t} = 3\cdot 0 = 0\end{aligned}$$

and so we can consider the point $(x,y) = (0,0)$ to be a horizontal point.

6.42 Example: Find the area of the region R which lies inside the limaçon $r = 1+2\cos\theta$ with $1 \le x \le 2$.

Solution: The limçon can be given parametrically in cartesian coordinates by

$$(x, y) = \big((1+2\cos t)\cos t, (1+2\cos t)\sin t\big)\,.$$

Note that since $x(t) = \cos t + 2\cos^2 t$ we have

$$x'(t) = -\sin t - 4\sin t\cos t = -\sin t\,(1+4\cos t)\,.$$

The top half of the region R is the region which lies below the portion of the limaçon with $0 \le t \le \frac{\pi}{3}$ and above the x-axis with $1 \le x \le 3$. Note that $y(t) \ge 0$ and $x'(t) \le 0$ for $0 \le t \le \frac{\pi}{3}$, and so the area of R is

$$A = -2\int_{t=0}^{\pi/3} y(t)\,x'(t)\,dt = 2\int_0^{\pi/3} (1+2\cos t)\cos t \cdot \sin t\,(1+4\cos t)\,dt\,.$$

We let $u = \cos t$ so $du = -\sin t\,dt$ to get

$$\begin{aligned} A &= \int_{u=1}^{1/2} -2u(1+2u)(1+4u)\,du = \int_{1/2}^{1} 2u + 12u^2 + 16u^3\;du \\ &= \Big[u^2 + 4u^3 + 4u^4\Big]_{1/2}^{1} = \big(1+4+4\big) - \big(\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4}\big) = 8\,. \end{aligned}$$

6.43 Example: Find a formula for the length of the polar curve $r = r(\theta)$ with $\alpha \le \theta \le \beta$.

Solution: The curve can be given parametrically, in cartesian coordinates, by

$$(x, y) = \big(r(t)\cos t, r(t)\sin t\big)$$

so we have

$$\begin{aligned} x'(t) &= r'(t)\cos t - r(t)\sin t \\ y'(t) &= r'(t)\sin t + r(t)\sin t \end{aligned}$$

and hence

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= \big(r'(t)^2\cos^2 t - 2\,r(t)r'(t)\sin t\cos t + r(t)^2\sin^2 t\big) \\ &\quad + \big(r'(t)^2\sin^2 t - 2\,r(t)r'(t)\sin t\cos t + r(t)^2\cos^2 t\big) \\ &= r'(t)^2 + r(t)^2\,. \end{aligned}$$

Thus the length of the curve is

$$L = \int_{t=\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\;dt = \int_{t=\alpha}^{\beta} \sqrt{r'(t)^2 + r(t)^2}\;dt\,.$$

6.44 Example: Find the length of the cardioid $r = 1 + \cos\theta$.

Solution: We have $r(t) = 1+\cos t$ and $r'(t) = -\sin t$. The top half of the cardioid is given by $0 \le t \le \pi$, and note that when $0 \le t \le \pi$ we have $\cos(t/2) \ge 0$. Using the above formula, the length of the cardioid is

$$\begin{aligned} L &= 2\int_{t=0}^{\pi} \sqrt{r(t)^2 + r'(t)^2}\;dt = 2\int_0^{\pi} \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}\;dt \\ &= \int_0^{\pi} 2\sqrt{2+2\cos t}\;dt = \int_0^{\pi} 2\sqrt{4\cos^2(t/2)}\;dt = \int_0^{\pi} 4\cos(t/2)\;dt \\ &= \Big[8\sin(t/2)\Big]_0^{\pi} = 8\,. \end{aligned}$$

6.45 Note: There is an alternative (and often preferable) way to calculate the area of a region which is described using polar coordinates. Consider the region R given in polar coordinates by $\alpha \leq \theta \leq \beta$, $f(\theta) \leq r \leq g(\theta)$. We can approximate the area of R as follows. Choose a partition $\alpha = \theta_1 < \theta_2 < \cdots < \theta_n = \beta$ of the interval $[\alpha, \beta]$. Choose sample points $c_i \in [\theta_{i-1}, \theta_i]$. Slice the region R into thin wedges with the i^{th} wedge given by $\theta_{i-1} \leq \theta \leq \theta_i$, $f(\theta) \leq r \leq g(\theta)$. The area of the i^{th} wedge is approximately

$$\Delta_i A \cong \tfrac{1}{2}\big(g(c_i)^2 - f(c_i)^2\big)\Delta_i \theta$$

where $\Delta_i \theta = \theta_i - \theta_{i-1}$. The total area is approximately

$$A \cong \sum_{i=1}^{n} \tfrac{1}{2}\left(g(c_i)^2 i f(c_i)^2\right)\Delta_i \theta\,.$$

The sum is a Riemann sum for the function $\frac{1}{2}\big(g(\theta)^2 - f(\theta)^2\big)$ on the interval $[\alpha, \beta]$, and so the exact area of R is the limit of these Riemann sums, that is

$$A = \int_{\theta=\alpha}^{\beta} \tfrac{1}{2}\big(g(\theta)^2 - f(\theta)^2\big)\, d\theta\,.$$

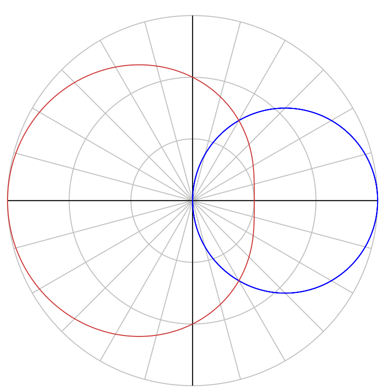
6.46 Example: Find the area of the region R which lies inside the cardioid $r = 1 + \cos\theta$.

Solution: Using the above formula, the area is

$$\begin{aligned} A &= \int_{\theta=0}^{2\pi} \tfrac{1}{2}\big(1+\cos\theta\big)^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2} + \cos\theta + \tfrac{1}{2}\cos^2\theta\ d\theta \\ &= \int_0^{2\pi} \tfrac{1}{2} + \cos\theta + \tfrac{1}{4} + \tfrac{1}{4}\cos 2\theta\ d\theta = \int_0^{2\pi} \tfrac{3}{4} + \cos\theta + \tfrac{1}{4}\cos 2\theta\ d\theta \\ &= \Big[\tfrac{3}{4}\,\theta + \sin\theta + \tfrac{1}{8}\sin 2\theta\Big]_0^{2\pi} = \tfrac{3\pi}{2}\,. \end{aligned}$$

6.47 Example: Find the area of the region R which lies inside both the circle $r = 3\cos\theta$ and the limaçon $r = 2 - \cos\theta$.

Solution: First we make a sketch (by plotting points). The curve $r = 3\cos\theta$ is shown in blue (it is a circle) and the curve $r = 2 - \cos\theta$ is shown in red.



The sketch helps to set up the integral. The total area is twice the area of the portion above the x-axis, which we divide into the portion with $0 \leq \theta \leq \frac{\pi}{3}$ and the portion with

$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$. The total area is

$$
\begin{aligned}
A &= 2\left(\int_0^{\pi/3} \tfrac{1}{2}(2-\cos\theta)^2 \, d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}(3\cos\theta)^2 \, d\theta\right) \\
&= \int_0^{\pi/3} \left(4 - 4\cos\theta + \cos^2\theta\right) d\theta + \int_{\pi/3}^{\pi/2} 9\cos^2\theta \; d\theta \\
&= \int_0^{\pi/3} \left(4 - 4\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right) d\theta + \int_{\pi/3}^{\pi/2} \tfrac{9}{2}(1+\cos 2\theta)\, d\theta \\
&= \left[\tfrac{9}{2}\theta - 4\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/3} + \left[\tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta\right]_{\pi/3}^{\pi/2} \\
&= \left(\tfrac{3\pi}{2} - 2\sqrt{3} + \tfrac{\sqrt{3}}{8}\right) + \left(\tfrac{9\pi}{4}\right) - \left(\tfrac{3\pi}{2} + \tfrac{9\sqrt{3}}{8}\right) \\
&= \tfrac{9\pi}{4} - 3\sqrt{3}
\end{aligned}
$$