Chapter 7. Sequences and Series

Review of Sequences

7.1 Definition: A **sequence** (of real numbers) is a function $a : \{k, k+1, k+2, \cdots\} \to \mathbb{R}$ for some integer k. For a sequence $a : \{k, k+1, \cdots\} \to \mathbb{R}$, we write $a_n = a(n)$ for $n \geq k$, we refer to the function a as the sequence $\{a_n\}$ or the sequence $\{a_n\}_{n \geq k}$, and we write

$$\{a_n\}_{n \geq k} = a_k, a_{k+1}, a_{k+2}, \cdots$$

We say that $\{a_n\}_{n \geq k}$ lies in the set $I \subset \mathbb{R}$ when $a_n \in I$ for every $n \geq k$.

7.2 Definition: We say the sequences $\{a_n\}_{n \geq k}$ **converges** to the real number l, or that the **limit** of the sequence $\{a_n\}_{n \geq k}$ is equal to l, and we write $\lim\limits_{n \to \infty} a_n = l$ or we write $a_n \to l$ (as $n \to \infty$), when for every $\epsilon > 0$ there exists $N \geq k$ such that for every integer n we have

$$n > N \Longrightarrow |a_n - l| < \epsilon \,.$$

We say the sequence $\{a_n\}$ **converges** if it converges to some real number l.

We say the limit of $\{a_n\}$ is equal to infinity, and write $\lim\limits_{n \to \infty} a_n = \infty$ or $a_n \to \infty$ when for every $R \in \mathbb{R}$ there exists $N \geq k$ such that for every integer n we have

$$n > N \Longrightarrow a_n > R \,.$$

We say the limit of $\{a_n\}$ is equal to negative infinity and write $\lim\limits_{n \to \infty} a_n = -\infty$ or $a_n \to -\infty$ when for every $R \in \mathbb{R}$ there exists $N \geq k$ such that for every integer n we have

$$n > N \Longrightarrow a_n < R \,.$$

7.3 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let l be a positive integer. Then $\lim\limits_{n \to \infty} a_n$ exists if and only if $\lim\limits_{n \to \infty} a_{n+l}$ exists, and in this case the limits are equal.*

7.4 Theorem: *(Linearity, Products and Quotients) If $\{a_n\}$ and $\{b_n\}$ are convergent sequences then*
(1) for any real number c, $\{c\, a_n\}$ converges with $\lim\limits_{n \to \infty} c\, a_n = c \lim\limits_{n \to \infty} a_n$,
(2) the sequence $\{a_n + b_n\}$ converges with $\lim\limits_{n \to \infty} (a_n + b_n) = \lim\limits_{n \to \infty} a_n + \lim\limits_{n \to \infty} b_n$,
(3) the sequence $\{a_n b_n\}$ converges with $\lim\limits_{n \to \infty} (a_n b_n) = \left(\lim\limits_{n \to \infty} a_n\right)\left(\lim\limits_{n \to \infty} b_n\right)$, and
(4) if $\lim\limits_{n \to \infty} b_n \neq 0$ then the sequence $\left\{\dfrac{a_n}{b_n}\right\}$ converges with $\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n \to \infty} a_n}{\lim\limits_{n \to \infty} b_n}$.

7.5 Note: By defining algebraic operations on the extended real line $\mathbb{R} \cup \{\pm\infty\}$, the above theorem can be extended to include many cases in which $\lim\limits_{n \to \infty} a_n = \pm\infty$ or $\lim\limits_{n \to \infty} b_n = \pm\infty$, but some care is needed for the so called **indeterminate forms** $\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$.

7.6 Theorem: *(Sequences and Functions) Let* $f : [a, \infty) \to \mathbb{R}$ *be a function and let* $a_n = f(n)$ *for all integers* $n \geq k$*. If* $\lim_{x\to\infty} f(x) = l$ *then* $\lim_{n\to\infty} a_n = l$ *(where the limit* l *can be finite or infinite) .*

7.7 Example: Find $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

Solution: Let $f(x) = \left(1 + \frac{1}{x}\right)^x = e^{x \ln(1+1/x)}$ so that $a_n = f(n)$ for all $n \geq 1$. By l'Hôpital's Rule we have

$$\lim_{x\to\infty} x \ln\left(1 + \tfrac{1}{x}\right) = \lim_{x\to\infty} \frac{1 + \frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = 1$$

and so $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = e^1 = e$.

7.8 Theorem: *(Comparison and Squeeze) Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be sequences.*
(1) If $a_n \leq b_n$ *for all* n *and* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *both exist, then* $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$
(2) If $a_n \leq b_n \leq c_n$ *for all* $n \geq k$ *and if* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ *then* $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$*.*

7.9 Theorem: *(Sequences and Absolute Values) Let* $\{a_n\}$ *be a sequence.*
(1) If $\lim_{n\to\infty} a_n$ *exists then* $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right|$*.*
(2) If $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$*.*
(3) If $|a_n| \leq b_n$ *for all* $n \geq k$ *and* $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$*.*

7.10 Definition: The sequence $\{a_n\}_{n\geq k}$ is called **increasing** when $a_n \leq a_{n+1}$ for all $n \geq k$, or equivalently when $n \leq m \Longrightarrow a_n \leq a_m$ for all integers $n, m \geq k$. The sequence $\{a_n\}_{n\geq k}$ is called **strictly increasing** when $a_n < a_{n+1}$ for all $n \geq k$. The sequence $\{a_n\}_{n\geq k}$ is **bounded above** by the real number b when $a_n \leq b$ for all $n \geq k$, and in this case b is called an **upper bound** for the sequence. We say that $\{a_n\}$ is **bounded above** when it is bounded above by some real number b. We have similar definitions for the terms **decreasing**, **strictly decreasing**, **bounded below** and **lower bound**.

7.11 Theorem: *(Monotone Convergence)*
(1) If $\{a_n\}$ *is increasing and bounded above by* b*, then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n \leq b$*.*
(2) If $\{a_n\}$ *is increasing and is not bounded above, then* $\lim_{n\to\infty} a_n = \infty$*.*
(3) If $\{a_n\}$ *is decreasing and bounded below by* c*, then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n \geq c$*.*
(4) If $\{a_n\}$ *is decreasing and is not bounded below, then* $\lim_{n\to\infty} a_n = -\infty$*.*

7.12 Example: Let $a_1 = 2$ and for $n \geq 1$ let $a_{n+1} = \dfrac{{a_n}^2 + 3}{4}$. Show that $\{a_n\}$ converges, and find the limit.

Solution: Suppose, for the moment, that $\{a_n\}$ converges and let $l = \lim\limits_{n\to\infty} a_n$. Note that we also have $\lim\limits_{n\to\infty} a_{n+1} = l$ (by Theorem 6.3) and so by taking the limit on both sides of the recurrence equation $a_{n+1} = \frac{a_n+3}{4}$ we find that $l = \frac{l^2+3}{4}$, that is $4l = l^2 + 3$, so we have $0 = l^2 - 4l + 3 = (l-1)(l-3)$ and so $l = 1$ or 3. This argument shows that if $\{a_n\}$ converges then the limit must be 1 or 3.

We claim that $1 \leq a_{n+1} \leq a_n \leq 3$ for all $n \geq 1$. When $n = 1$ we have $a_n = a_1 = 2$ and $a_{n+1} = a_2 = \frac{7}{4}$ and so the claim is true when $n = 1$. Fix $k \geq 1$ and suppose, inductively, that the claim is true when $n = k$. Then we have

$$1 \leq a_{k+1} \leq a_k \leq 3 \Longrightarrow 1 \leq {a_{k+1}}^2 \leq {a_k}^2 \leq 9 \Longrightarrow 4 \leq {a_{k+1}}^2 + 3 \leq {a_k}^2 + 3 \leq 12$$
$$\Longrightarrow 1 \leq \tfrac{{a_{k+1}}^2+3}{4} \leq \tfrac{{a_k}^2+3}{4} \leq 3 \Longrightarrow 1 \leq a_{k+2} \leq a_{k+1} \leq 3$$

and so the claim is also true when $n = k+1$. By Mathematical Induction, the claim is true for all $n \geq 1$. Thus we have $1 \leq a_{n+1} \leq a_n \leq 3$ for all $n \geq 1$

Since $a_{n+1} \leq a_n$ for all n, the sequence is decreasing, and since $1 \leq a_n$ for all n, the sequence is bounded below. By the Monotone Convergence Theorem, the sequence does converge. Since we know that the limit must be 1 or 3, and since the sequence starts at 2 and decreases, it follows that the limit must be 1. Thus $\lim\limits_{n\to\infty} a_n = 1$.

7.13 Definition: Let $\{a_n\}_{n\geq k}$ be a sequence. The **series** $\sum_{n\geq k} a_n$ is defined to be the sequence $\{S_l\}_{l\geq k}$ where

$$S_l = \sum_{n=k}^{l} a_n = a_k + a_{k+1} + \cdots + a_l \,.$$

The term S_l is called the l^{th} **partial sum** of the series $\sum_{n\geq k} a_n$. The **sum** of the series, denoted by

$$S = \sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots \,,$$

is the limit of the sequence of partial sums, if it exists, and we say the series **converges** when the sum exists and is finite.

7.14 Example: (Geometric Series) Show that for $a \neq 0$, the series $\sum_{n\geq k} a_n$ converges if and only if $|r| < 1$, and that in this case

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r} \,.$$

Solution: The l^{th} partial sum is

$$S_l = \sum_{n=k}^{\infty} ar^n = ar^k + ar^{k+1} + ar^{k+2} + \cdots + ar^l \,.$$

When $r = 1$ we have $S_l = a(l-k+1)$ and so $\lim_{l\to\infty} S_l = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \neq 1$ we have $rS_l = ar^{k+1} + ar^{k+2} + \cdots + ar^l + ar^{l+1}$, so $S_l - rS_l = ar^k - ar^{l+1} = ar^k\big(1 - r^{l-k+1}\big)$ and so

$$S_l = \frac{ar^k(1 - r^{l-k+1})}{1-r} \,.$$

When $r > 1$, $\lim_{l\to\infty} r^{l-k+1} = \infty$ and so $\lim_{l\to\infty} S_l = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \leq -1$, $\lim_{l\to\infty} r^{l-k+1}$ does not exist, and so neither does $\lim_{l\to\infty} S_l$. When $|r| < 1$, we have $\lim_{l\to\infty} r^{l-k+1} = 0$ and so $\lim_{l\to\infty} S_l = \dfrac{ar^k}{1-r}$, as required.

7.15 Example: Find $\displaystyle\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}}$.

Solution: This is a geometric series. By the formula in the previous example, we have

$$\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}} = \sum_{n=-1}^{\infty} \frac{3\cdot 3^n}{2^{-1}\cdot 4^n} = \sum_{n=-1}^{\infty} 6\left(\tfrac{3}{4}\right)^n = \frac{6\left(\frac{3}{4}\right)^{-1}}{1-\frac{3}{4}} = \frac{6\cdot\frac{4}{3}}{\frac{1}{4}} = 32 \,.$$

7.16 Example: (Telescoping Series) Find $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^2+2n}$.

Solution: We use a partial fractions decomposition. The l^{th} partial sum is

$$\begin{aligned} S_l &= \sum_{n=1}^{l} \frac{1}{n(n+2)} = \sum_{n=1}^{l} \left(\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} \right) = \tfrac{1}{2} \sum_{n=1}^{l} \left(\tfrac{1}{n} - \tfrac{1}{n+2} \right) \\ &= \tfrac{1}{2} \Big(\big(1 - \tfrac{1}{3}\big) + \big(\tfrac{1}{2} - \tfrac{1}{4}\big) + \big(\tfrac{1}{3} - \tfrac{1}{5}\big) + \cdots + \left(\tfrac{1}{n-2} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n-1} - \tfrac{1}{n+1}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+2}\right) \Big) \\ &= \tfrac{1}{2} \left(1 + \tfrac{1}{2} - \tfrac{1}{n+1} - \tfrac{1}{n+2} \right) , \end{aligned}$$

since all the other terms cancel. Thus the sum of the series is

$$S = \lim_{l \to \infty} S_l = \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right) = \tfrac{3}{4} .$$

7.17 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let* $\{a_n\}_{n \geq k}$ *be a sequence. Then for any integer* $m \geq k$*, the series* $\displaystyle\sum_{n \geq k} a_n$ *converges if and only if the series* $\displaystyle\sum_{n \geq m} a_n$ *converges, and in this case*

$$\sum_{n=k}^{\infty} a_n = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \sum_{n=m}^{\infty} a_n .$$

Proof: Let $\displaystyle S_l = \sum_{n=k}^{l} a_n$ and let $\displaystyle T_l = \sum_{n=m}^{l} a_n$. Then for all $l \geq m$ we have

$$S_l = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + T_l ,$$

and so $\{S_l\}$ converges if and only if $\{T_l\}$ converges, and in this case

$$\lim_{l \to \infty} S_l = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \lim_{l \to \infty} T_l .$$

7.18 Note: Since the first finitely many terms do not affect the convergence of a series, we often omit the subscript $n \geq k$ in the expression $\displaystyle\sum_{n \geq k} a_n$ when we are interested in whether or not the series converges. On the other hand, we cannot omit the subscript $n = k$ when we are interested in the value of the sum $\displaystyle\sum_{n=k}^{\infty} a_n$.

7.19 Definition: When we approximate a value x by the value y, the (absolute) **error** in our approximation is $|x - y|$.

7.20 Note: If $\displaystyle\sum_{n \geq k} a_n$ converges and $l \geq k$ then, by the above theorem, so does $\displaystyle\sum_{n \geq l+1}^{\infty} a_n$.

If we approximate the sum $\displaystyle S = \sum_{n=k}^{\infty} a_n$ by the l^{th}partial sum $\displaystyle S_l = \sum_{n=k}^{l} a_n$, then the **error** in our approximation is

$$\big|S - S_l\big| = \left| \sum_{n=l+1}^{\infty} a_n \right| .$$

7.21 Theorem: *(Linearity) If* $\sum a_n$ *and* $\sum b_n$ *are convergent series then*

(1) for any real number c, $\sum ca_n$ *converges and* $\displaystyle\sum_{n=k}^{\infty} ca_n = c\sum_{n=k}^{\infty} a_n$ *, and*

(2) the series $\sum(a_n + b_n)$ *converges and* $\displaystyle\sum_{n=k}^{\infty}(a_n + b_n) = \sum_{n=k}^{\infty} a_n + \sum_{n=k}^{\infty} b_n$ *.*

Proof: This follows immediately from the Linearity Theorem for sequences.

7.22 Theorem: *(Series of Positive Terms) Let* $\sum a_n$ *be a series.*

(1) If $a_n \geq 0$ *for all* $n \geq k$ *then either* $\sum a_n$ *converges or* $\displaystyle\sum_{n=k}^{\infty} a_n = \infty$*.*

(2) If $a_n \leq 0$ *for all* $n \geq k$ *then either* $\sum a_n$ *converges or* $\displaystyle\sum_{n=k}^{\infty} a_n = -\infty$*.*

Proof: This follows from the Monotone Convergence Theorem for sequences. Indeed if $a_n \geq 0$ for all $n \geq k$, then $\{S_l\}$ is increasing (since $S_{l+1} = S_l + a_{l+1} \geq S_l$ for all l). Either $\{S_l\}$ is bounded above, in which case $\{S_l\}$ converges hence $\sum a_n$ converges, or $\{S_l\}$ is unbounded, in which case $\displaystyle\lim_{n\to\infty} S_l = \infty$ hence $\displaystyle\sum_{n=k}^{\infty} a_n = \infty$.

7.23 Theorem: *(Divergence Test) If* $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$. *Equivalently, if* $\lim_{n\to\infty} a_n$ *either does not exist, or exists but is not equal to 0, then* $\sum a_n$ *diverges.*

Proof: Suppose that $\sum a_n$ converges, and say $\sum_{n=k}^{\infty} a_n = S$. Let S_l be the l^{th}partial sum. Then $\lim_{l\to\infty} S_l = S = \lim_{l\to\infty} S_{l-1}$, and we have $a_l = S_l - S_{l-1}$, and so

$$\lim_{l\to\infty} a_l = \lim_{l\to\infty} S_l - \lim_{l\to\infty} S_{l-1} = S - S = 0\,.$$

7.24 Example: Determine whether $\sum e^{1/n}$ converges.

Solution: Since $\lim_{n\to\infty} e^{1/n} = e^0 = 1$, $\sum e^{1/n}$ diverges by the Divergence Test.

7.25 Note: The converse of the Divergence Test is false. For example, as we shall see in Example 6.27 below, $\sum \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

7.26 Theorem: *(Integral Test) Let* $f(x)$ *be positive and decreasing for* $x \geq k$, *and let* $a_n = f(n)$ *for all integers* $n \geq k$. *Then* $\sum a_n$ *converges if and only if* $\int_k^\infty f(x)\,dx$ *converges, and in this case, for any* $l \geq k$ *we have*

$$\int_{l+1}^\infty f(x)\,dx \leq \sum_{n=l+1}^{\infty} a_n \leq \int_l^\infty f(x)\,dx\,.$$

Proof: Let T_m be the m^{th} partial sum for $\sum_{n\geq l+1} a_n$, so $T_m = \sum_{n=l+1}^{m} a_n$. Note that since $f(x)$ is decreasing, it is integrable on any closed interval. Also, for each $n \geq l$ we have $a_n = f(n) \leq f(x)$ for all $x \in [n-1, n]$, so $\int_{n-1}^{n} f(x)\,dx \geq \int_{n-1}^{n} a_n\,dx = a_n$ and so

$$T_m = \sum_{n=l+1}^{m} a_n \leq \sum_{n=l+1}^{m} \int_{n-1}^{n} f(x)\,dx = \int_l^m f(x)\,dx \leq \int_l^\infty f(x)\,dx\,.$$

Since $f(n) = a_n$ is positive, the sequence $\{T_m\}$ is increasing. If $\int_k^\infty f$ converges, then $\{T_n\}$ is bounded above by $\int_l^\infty f(x)\,dx$, and so it converges with $\lim_{m\to\infty} T_m \leq \int_l^\infty f(x)\,dx$.

Similarly, for each $n \geq l$ we have $a_n = f(n) \geq f(x)$ for all $x \in [n, n+1]$ so that $\int_n^{n+1} f(x)\,dx \leq \int_n^{n+1} a_n dx = a_n$ and so

$$T_m = \sum_{n=l+1}^{m} a_n \geq \sum_{n=l+1}^{m} \int_n^{n+1} f(x)\,dx = \int_{l+1}^{m+1} f(x)\,dx\,.$$

If $\int_k^\infty f$ converges, then $\lim_{m\to\infty} T_m \geq \lim_{m\to\infty} \int_{l+1}^{m+1} f(x)\,dx = \int_{l+1}^\infty f(x)\,dx$. If $\int_k^\infty f = \infty$ then $\lim_{m\to\infty} \int_{l+1}^{m+1} f(x)\,dx = \infty$, and so $\lim_{m\to\infty} T_m = \infty$ too, by Comparison.

7.27 Example: (p-Series) Show that the series $\displaystyle\sum_{n\geq 1}\frac{1}{n^p}$ converges if and only if $p>1$. In particular, the **harmonic series** $\sum\frac{1}{n}$ diverges.

Solution: If $p<0$ then $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}=\infty$ and if $p=0$ then $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}=1$, so in either case $\sum\frac{1}{n^p}$ diverges by the Divergence Test. Suppose that $p>0$. Let $a_n=\frac{1}{n^p}$ for integers $n\geq 1$, and let $f(x)=\frac{1}{x^p}$ for real numbers $x\geq 1$. Note that $f(x)$ is positive and decreasing for $x\geq 1$ and $a_n=f(n)$ for all $n\geq 1$. Since we know that $\displaystyle\int_1^\infty f(x)\,dx$ converges if and only if $p>1$, it follows from the Integral Test that $\sum a_n$ converges if and only if $p>1$.

7.28 Example: Approximate $S=\displaystyle\sum_{n=1}^\infty\frac{1}{2n^2}$ so that the error is at most $\frac{1}{100}$.

Solution: We let $a_n=\frac{1}{2n^2}$ and $f(x)=\frac{1}{2x^2}$ so that we can apply the Integral Test. If we choose to approximate the sum S by the l^{th}partial sum S_l, then the error is

$$E=S-S_l=\sum_{n=l+1}^\infty a_n\leq\int_l^\infty\frac{1}{2x^2}\,dx=\Big[-\frac{1}{2x}\Big]_l^\infty=\frac{1}{2l}\,,$$

and so to insure that $E\leq\frac{1}{100}$ we can choose l so that $\frac{1}{2l}\leq\frac{1}{100}$, that is $l\geq 50$. Since it would be tedious to add up the first 50 terms of the series, we take an alternate approach. The Integral Test gives us upper and lower bounds: we have

$$\begin{gathered}\int_{l+1}^\infty f(x)\,dx\leq S-S_l\leq\int_l^\infty f(x)\,dx\\ \frac{1}{2(l+1)}\leq S-S_l\leq\frac{1}{2l}\\ S_l+\frac{1}{2(l+1)}\ \leq S\leq\ S_l+\frac{1}{2l}\,.\end{gathered}$$

If approximate S using the midpoint of the upper and lower bounds, that is if we make the approximation $S\cong S_l+\frac{1}{2}\left(\frac{1}{2l}+\frac{1}{2(l+1)}\right)$, then the error E will be at most half of the difference of the bounds:

$$E\leq\tfrac{1}{2}\left(\tfrac{1}{2l}-\tfrac{1}{2(l+1)}\right)=\tfrac{1}{4l(l+1)}\,.$$

To get $E\leq\frac{1}{100}$ we want $\frac{1}{4\,l\,(l+1)}\leq\frac{1}{100}$, that is $l\,(l+1)\geq 25$, and so we can take $l=5$. Thus we estimate

$$S\cong S_5+\tfrac{1}{2}\left(\tfrac{1}{10}+\tfrac{1}{12}\right)=\tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{18}+\tfrac{1}{32}+\tfrac{1}{50}+\tfrac{1}{20}+\tfrac{1}{24}=\tfrac{5929}{7200}\,.$$

(Incidentally, the exact value of this sum is $\frac{\pi^2}{12}$).

7.29 Theorem: *(Comparison Test) Let $0 \le a_n \le b_n$ for all $n \ge k$. Then if $\sum b_n$ converges then so does $\sum a_n$ and in this case,*

$$\sum_{n=k}^{\infty} a_n \le \sum_{n=k}^{\infty} b_n \,.$$

Proof: Let $S_l = \sum_{n=k}^{l} a_n$ and let $T_l = \sum_{n=k}^{l} b_n$. Since $0 \le a_n, b_n$ for all n, the sequences $\{S_l\}$ and $\{T_l\}$ are increasing. Since $a_n \le b_n$ for all n we have $S_l \le T_l$ for all l. Suppose that $\sum b_n$ converges with say $\sum_{n=k}^{\infty} b_n = T$ so that $\lim_{l\to\infty} \{T_l\} = T$. Then $S_l \le T_l \le T$ for all l, so $\{S_l\}$ is increasing and bounded above, hence convergent, and $\lim_{l\to\infty} S_l \le \lim_{l\to\infty} T_l$.

7.30 Example: Determine whether $\displaystyle\sum_{n\ge 0} \frac{1}{\sqrt{n^3+1}}$ converges.

Solution: Note that $0 \le \frac{1}{\sqrt{n^3+1}} \le \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$ for all $n \ge 1$, and $\sum \frac{1}{n^{3/2}}$ converges since it is a p-series with $p = \frac{3}{2}$, and so $\sum \frac{1}{\sqrt{n^3+1}}$ also converges, by the Comparison Test.

7.31 Example: Determine whether $\displaystyle\sum_{n\ge 1} \tan \frac{1}{n}$ converges.

Solution: For $0 < x < \frac{\pi}{2}$ we have $x < \tan x$, so for $n \ge 1$ we have $0 < \frac{1}{n} < \tan \frac{1}{n}$. Since the harmonic series $\sum \frac{1}{n}$ diverges, the series $\sum \tan \frac{1}{n}$ also diverges by the Comparison Test.

7.32 Example: Approximate $S = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ so that the error is at most $\frac{1}{100}$.

Solution: If we make the approximation $S \cong S_l = \displaystyle\sum_{n=0}^{l} \frac{1}{n!}$ then the error is

$$\begin{aligned} E = S - S_l &= \sum_{n=l+1}^{\infty} \frac{1}{n!} \\ &= \tfrac{1}{(l+1)!} + \tfrac{1}{(l+2)!} + \tfrac{1}{(l+3)!} + \tfrac{1}{(l+4)!} + \cdots \\ &= \tfrac{1}{(l+1)!} \left(1 + \tfrac{1}{l+2} + \tfrac{1}{(l+2)(l+3)} + \tfrac{1}{(l+2)(l+3)(l+4)} + \cdots\right) \\ &\le \tfrac{1}{(l+1)!} \left(1 + \tfrac{1}{l+2} + \tfrac{1}{(l+2)^2} + \tfrac{1}{(l+2)^3} + \cdots\right) \\ &= \tfrac{1}{(l+1)!} \frac{1}{1 - \frac{1}{l+2}} \\ &= \tfrac{l+2}{(l+1)(l+1)!} \end{aligned}$$

where we used the Comparison Test and the formula for the sum of a geometric series. To get $E \le \frac{1}{100}$ we can choose l so that $\frac{l+2}{(l+1)(l+1)!} \le \frac{1}{100}$. By trial and error, we find that we can take $l = 4$, so we make the approximation

$$S \cong S_4 = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} = \tfrac{65}{24} \,.$$

(Incidentally, we shall see later that the exact value of this sum is e).

7.33 Theorem: *(Limit Comparison Test) Let $a_n \geq 0$ and let $b_n > 0$ for all $n \geq k$. Suppose that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = r$. Then*
(1) if $r = \infty$ and $\sum a_n$ converges then so does $\sum b_n$,
(2) if $r = 0$ and $\sum b_n$ converges then so does $\sum a_n$, and
(3) if $0 < r < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof: If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, then for large n we have $\frac{a_n}{b_n} > 1$ so that $a_n > b_n$, and so if $\sum a_n$ converges, then so does $\sum b_n$ by the Comparison Test. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ then for large n we have $\frac{a_n}{b_n} < 1$ so $a_n < b_n$, and so if $\sum b_n$ converges then so does $\sum a_n$ by the Comparison Test. Suppose that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = r$ with $0 < r < \infty$. Choose N so that when $n > N$ we have $\left|\frac{a_n}{b_n} - r\right| < \frac{r}{2}$ so that $\frac{r}{2} < \frac{a_n}{b_n} < \frac{3r}{2}$ and hence

$$0 < \tfrac{r}{2} b_n \leq a_n \leq \tfrac{3r}{2} b_n \,.$$

If $\sum a_n$ converges, then $\sum \frac{r}{2} b_n$ converges by the Comparison Test, and hence $\sum b_n$ converges by linearity. If $\sum b_n$ converges, then $\sum \frac{3r}{2} b_n$ converges by linearity, and hence so does $\sum a_n$ by the Comparison Test.

7.34 Example: Determine whether $\sum \frac{1}{\sqrt{n^3-1}}$ converges.

Solution: Note that we cannot use the same argument that we used earlier to show that $\sum \frac{1}{\sqrt{n^3+1}}$ converges, because $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$ but $\frac{1}{\sqrt{n^3-1}} > \frac{1}{n^{3/2}}$. We use a different approach. Let $a_n = \frac{1}{\sqrt{n^3-1}}$ and let $b_n = \frac{1}{n^{3/2}}$. Then $\lim \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{\sqrt{n^3-1}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1-\frac{1}{n^3}}} = 1$, and $\sum b_n = \sum \frac{1}{n^{3/2}}$ converges (its a p-series with $p = \frac{3}{2}$), and so $\sum a_n$ converges too, by the Limit Comparison Test.

7.35 Theorem: *(Ratio Test) Let $a_n > 0$ for all $n \geq k$. Suppose $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$. Then*
(1) if $r < 1$ then $\sum a_n$ converges, and
(2) if $r > 1$ then $\lim\limits_{n\to\infty} a_n = \infty$ so $\sum a_n = \infty$.

Proof: Suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. Choose s with $r < s < 1$, and then choose N so that when $n > N$ we have $\frac{a_{n+1}}{a_n} < s$ and hence $a_{n+1} < s\, a_n$. Fix $k > N$. Then $a_{k+1} < sa_k$, $a_{k+2} < sa_{k+1} < s^2 a_k$, $a_{k+3} < sa_{k+2} < s^3 a_k$, and so on, so we have $a_n < b_n = s^{n-k} a_k$ for all $n \geq k$. Since $\sum b_n$ is geometric with ratio $s < 1$, it converges, and hence so does $\sum a_n$ by the Comparison Test.

Now suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$. Choose s with $1 < s < r$, then choose N so that when $n > N$ we have $\frac{a_{n+1}}{a_n} > s$ and hence $a_{n+1} > sa_n$. Fix $k > N$. Then as above $a_n > b_n = s^{n-k} a_k$ for all $n \geq k$, and $\lim\limits_{n\to\infty} b_n = \infty$, so $\lim\limits_{n\to\infty} a_n = \infty$ too.

7.36 Example: Determine whether $\sum \frac{5^n}{n!}$ converges.

Solution: Let $a_n = \frac{5^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \to 0$ as $n \to \infty$, and so $\sum a_n$ converges by the Ratio Test.

7.37 Note: If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then $\sum a_n$ could converge or diverge. For example, if $a_n = \frac{1}{n}$ then $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n \to \infty$ and $\sum a_n$ diverges, but if $b_n = \frac{1}{n^2}$ then $\frac{b_{n+1}}{b_n} = \frac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$ and $\sum b_n$ converges.

7.38 Theorem: *(Root Test) Let $a_n \geq 0$ for all $n \geq k$. Suppose that $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = r$. Then*
(1) if $r < 1$ then $\sum a_n$ converges, and
(2) if $r > 1$ then $\lim\limits_{n\to\infty} a_n = \infty$ so $\sum a_n = \infty$.

Proof: The proof is left as an exercise. It is similar to the proof of the Ratio Test.

7.39 Example: Determine whether $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges.

Solution: Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then $\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = e^{n \ln\left(\frac{n}{n+1}\right)}$, and by l'Hôpital's Rule we have $\lim\limits_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim\limits_{x\to\infty} \dfrac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{(x+1)^2}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{-x^2}{(x+1)^2} = -1$, and so $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1$. Thus $\sum a_n$ converges by the Root Test.

7.40 Definition: A sequence $\{a_n\}_{n\geq k}$ is said to be **alternating** when either we have $a_n = (-1)^n|a_n|$ for all $n \geq k$ or we have $a_n = (-1)^{n+1}|a_n|$ for all $n \geq k$.

7.41 Theorem: *(Aternating Series Test) Let $\{a_n\}_{n\geq k}$ be an alternating series. If $\{|a_n|\}$ is decreasing with $\lim\limits_{n\to\infty} |a_n| = 0$ then $\sum\limits_{n\geq k} a_n$ converges, and in this case*

$$\left|\sum_{n=k}^{\infty} a_n\right| \leq |a_k|\,.$$

Proof: To simplify notation, we give the proof in the case that $k = 0$ and $a_n = (-1)^n|a_n|$. Suppose that $\big\{|a_n|\big\}$ is decreasing with $|a_n| \to 0$. Let $S_l = \sum\limits_{n=0}^{l} a_n$. We consider the sequences $\{S_{2l}\}$ and $\{S_{2l-1}\}$ of even and odd partial sums. Note that since $\big\{|a_n|\big\}$ is decreasing, we have

$$S_{2l} - S_{2l-1} = |a_{2l}| - |a_{2l-1}| \leq 0$$

so $\big\{S_{2l}\big\}$ is decreasing, and we have

$$\begin{aligned} S_{2l} &= |a_0| - |a_1| + |a_2| - |a_3| + \cdots + |a_{2l-2}| - |a_{2l-1}| + |a_{2l}| \\ &= \big(|a_0| - |a_1|\big) + \big(|a_2| - |a_3|\big) + \cdots + \big(|a_{2l-2}| - |a_{2l-1}|\big) + |a_{2l}| \\ &\geq |a_0| - |a_1| \end{aligned}$$

and so $\{S_{2l}\}$ is bounded below by $|a_0| - |a_1|$. Thus $\big\{S_{2l}\big\}$ converges by the Monotone Convergence Theorem. Similarly, $\big\{S_{2l-1}\big\}$ is increasing and bounded above by $|a_0|$, so it also converges, and we have $\lim\limits_{l\to\infty} S_{2l-1} \leq |a_0|$.

Finally we note that since $|a_n| \to 0$, taking the limit on both sides of the equality $|a_{2l}| = S_{2l} - S_{2l-1}$ gives $0 = \lim\limits_{l\to\infty} S_{2l} - \lim\limits_{l\to\infty} S_{2l-1}$. and so we have $\lim\limits_{l\to\infty} S_{2l} = \lim\limits_{l\to\infty} S_{2l-1}$. It follows that $\big\{S_l\big\}$ converges, and we have $\lim\limits_{l\to\infty} S_l = \lim\limits_{l\to\infty} S_{2l} = \lim\limits_{l\to\infty} S_{2l-1} \leq |a_0|$.

7.42 Example: Determine whether $\sum\limits_{n\geq 2} \dfrac{(-1)^n \ln n}{\sqrt{n}}$ converges.

Solution: Let $a_n = \dfrac{(-1)^n \ln n}{\sqrt{n}}$. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ so that $|a_n| = f(n)$. Note that

$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{3/2}}\,,$$

so we have $f'(x) < 0$ for $x > e^2$. Thus $f(x)$ is decreasing for $x > e^2$, and so $\big\{|a_n|\big\}$ is decreasing for $n \geq 8$. Also, by l'Hôpital's Rule, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \tfrac{2}{\sqrt{x}} = 0$$

and so $|a_n| \to 0$ as $n \to \infty$. Thus $\sum a_n$ converges by the Alternating Series Test.

7.43 Example: Approximate the sum $S=\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}$ so that the error is at most $\frac{1}{2000}$.

Solution: Let $a_n=\frac{(-2)^n}{(2n)!}$. Note that

$$\frac{|a_{n+1}|}{|a_n|}=\frac{2^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{2^n}=\frac{2}{(2n+2)(2n+1)}=\frac{1}{(n+1)(2n+1)}\,.$$

Since $\frac{|a_{n+1}|}{|a_n|}\leq 1$ for all $n\geq 0$, we know that $\{|a_n|\}$ is decreasing. Since $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=0$, we know that $\sum|a_n|$ converges by the Ratio Test, and so $|a_n|\to 0$ by the Divergence Test. This shows that we can apply the Alternating Series Test.

If we approximate S by the l^{th}partial sum $S_l=\sum_{n=0}^{l}a_n$, then by the Alternating Series Test, the error is

$$E=\big|S-S_l\big|=\left|\sum_{n=l+1}^{\infty}a_n\right|\leq\big|a_{l+1}\big|=\frac{2^{l+1}}{(2l+2)!}\,.$$

To get $E\leq\frac{1}{2000}$ we can choose l so that $\frac{2^{l+1}}{(l+1)!}\leq\frac{1}{2000}$. By trial and error we find that we can take $l=3$. Thus we make the approximation

$$S\cong S_3=1-\tfrac{2}{2!}+\tfrac{2^2}{4!}-\tfrac{2^3}{6!}=1-1+\tfrac{1}{6}+\tfrac{1}{90}=\tfrac{7}{45}\,.$$

$\big($We shall see later that the exact value of this sum is $\cos\sqrt{2}\big)$.

7.44 Definition: A series $\sum_{n\geq k}a_n$ is said to **converge absolutely** when $\sum_{n\geq k}|a_n|$ converges. The series is said to **converge conditionally** if $\sum_{n\geq k}a_n$ converges but $\sum_{n\geq k}|a_n|$ diverges.

7.45 Example: For $0<p\leq 1$, the p-series $\sum\frac{1}{n^p}$ diverges, but since $\left\{\frac{1}{n^p}\right\}$ is decreasing towards 0, $\sum\frac{(-1)^n}{n^p}$ converges by the Alternating Series Test. Thus for $0<p\leq 1$, the alternating p-series $\sum\frac{(-1)^n}{n^p}$ converges conditionally.

7.46 Theorem: *(Absolute Convergence Implies Convergence) If $\sum|a_n|$ converges then so does $\sum a_n$.*

Proof: Suppose that $\sum|a_n|$ converges. Note that $-|a_n|\leq a_n\leq|a_n|$ so that

$$0\leq a_n+|a_n|\leq 2|a_n| \text{ for all } n\,.$$

Since $\sum|a_n|$ converges, $\sum 2|a_n|$ converges by linearity, and so $\sum\big(a_n+|a_n|\big)$ converges by the Comparison Test. Since $\sum|a_n|$ and $\sum\big(a_n+|a_n|\big)$ both converge, $\sum a_n$ converges by linearity.

7.47 Example: Determine whether $\sum\frac{\sin n}{n^2}$ converges.

Solution: Let $a_n=\frac{\sin n}{n^2}$. Then $|a_n|=\frac{|\sin n|}{n^2}\leq\frac{1}{n^2}$. Since $\sum\frac{1}{n^2}$ converges (its a p-series with $p=2$), $\sum|a_n|$ converges by the Comparison Test, and hence $\sum a_n$ converges too, since absolute convergence implies convergence.

Multiplication of Series

7.48 Theorem: *(Multiplication of Series) Suppose that* $\sum\limits_{n\geq 0}|a_n|$ *converges and* $\sum\limits_{n\geq 0} b_n$ *converges and define* $c_n=\sum\limits_{k=0}^{n} a_k b_{n-k}$. *Then* $\sum\limits_{n\geq 0} c_n$ *converges and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof: Let $A_l=\sum\limits_{n=0}^{l} a_n$, $B_l=\sum\limits_{n=0}^{l} b_n$, $C_l=\sum\limits_{n=0}^{l} c_n$, $A=\sum\limits_{n=0}^{\infty} a_n$, $B=\sum\limits_{n=0}^{\infty} b_n$, $K=\sum\limits_{n=0}^{\infty}|a_n|$ and $E_l=B-B_l$. Then we have

$$\begin{aligned}
C_l &= a_0b_0+(a_0b_1+a_1b_0)+(a_0b_2+a_1b_1+a_2b_0)+\cdots+(a_0b_l+\cdots+a_lb_0)\\
&= a_0B_l+a_1B_{l-1}+a_2B_{l-2}+\cdots+a_lB_0\\
&= a_0(B-E_l)+a_1(B-E_{l-1})+\cdots+a_l(B-E_0)\\
&= A_lB-\big(a_0E_l+a_1E_{l-1}+\cdots+a_lE_0\big)
\end{aligned}$$

and so

$$\big|AB-C_l\big| \leq \big|(A-A_l)B\big|+\big|a_0E_l+a_1E_{l-1}+\cdots+a_lE_0\big|.$$

Let $\epsilon>0$. Choose m so that $j>m \Longrightarrow E_j<\frac{\epsilon}{3K}$. Let $E=\max\big\{|E_0|,\cdots,|E_m|\big\}$. Choose $L>m$ so that when $l>L$ we have $\sum\limits_{n=l-m}^{l}|a_n|<\frac{\epsilon}{3E}$ and we have $|A_l-A||B|<\frac{\epsilon}{3}$. Then for $l>L$,

$$\begin{aligned}
\big|C_l-AB\big| &< \big|(A_l-A)B\big|+\big|a_0E_l+\cdots+a_{l-m-1}E_{m+1}\big|+\big|a_{l-m}E_m+\cdots+a_lE_0\big|\\
&\leq \tfrac{\epsilon}{3}+\left(\sum_{n=0}^{l-m-1}|a_n|\right)\tfrac{\epsilon}{3K}+\left(\sum_{n=l-m+1}^{l}|a_n|\right)E\\
&< \tfrac{\epsilon}{3}+K\tfrac{\epsilon}{3K}+\tfrac{\epsilon}{3E}E=\epsilon\,.
\end{aligned}$$

7.49 Example: Find an example of sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ such that $\sum\limits_{n\geq 0} a_n$ and $\sum\limits_{n\geq 0} b_n$ both converge, but $\sum\limits_{n\geq 0} c_n$ diverges where $c_n=\sum\limits_{k=0}^{n} a_kb_{n-k}$.

Solution: Let $a_n=b_n=\dfrac{(-1)^n}{\sqrt{n+1}}$ for $n\geq 0$, and let

$$c_n=\sum_{k=0}^{n} a_kb_{n-k}=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{(k+1)(n-k+1)}}\,.$$

Recall that for $p,q\geq 0$ we have $\sqrt{pq}\leq\frac{1}{2}(p+q)$ (indeed $(p+q)^2-4pq=p^2-2pq+q^2=(p-q)^2\geq 0$, so $(p+q)^2\geq 4pq$). In particular $\sqrt{(k+1)(n-k+1)}\leq\frac{1}{2}(n+2)$ and so $|c_n|\geq\sum\limits_{k=0}^{n}\frac{2}{n+2}=\frac{2(n+1)}{n+2}$. Thus $\lim\limits_{n\to\infty}|c_n|\neq 0$ so $\sum c_n$ diverges by the Divergence Test.