Chapter 8. Power Series

Power Series

8.1 Definition: A **power series centred at** a is a series of the form

$$\sum_{n\geq 0} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for some real numbers c_n, where we use the convention that $(x-a)^0 = 1$.

8.2 Example: The geometric series $\sum_{n\geq 0} x^n$ is a power series centred at 0. It converges when $|x| < 1$ and for all such x the sum of the series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

8.3 Theorem: *(The Interval and Radius of Convergence) Let* $\sum_{n\geq 0} c_n(x-a)^n$ *be a power series. Then the set of* $x\in\mathbb{R}$ *for which the power series converges is an interval* I *centred at* a*. Indeed there exists a (possibly infinite) number* $R\in[0,\infty]$ *such that*
(1) if $|x-a| < R$ *then* $\sum_{n\geq 0} c_n(x-a)^n$ *converges absolutely, and*
(2) if $|x-a| > R$ *then* $\sum_{n\geq 0} c_n(x-a)^n$ *diverges.*

Proof: We prove parts (1) and (2) together by showing that for all $r > 0$, if $\sum c_n r^n$ converges then $\sum c_n(x-a)^n$ converges absolutely for all $x \in R$ with $|x-a| < r$ (we can then take R to be the least upper bound of the set of all such r). Let $r > 0$. Suppose that $\sum c_n r^n$ converges. Let $x\in\mathbb{R}$ with $|x-a| < r$. Choose s with $|x-a| < s < r$. Since $\sum c_n r^n$ converges, we have $c_n r^n \to 0$ by the Divergence Test. Choose $N > 0$ so that $|c_n r^n| \leq 1$ for all $n \geq N$. Then for $n \geq N$ we have

$$\left|c_n(x-a)^n\right| = |c_n r^n|\cdot\frac{|x-a|^n}{r^n} \leq \frac{|x-a|^n}{r^n} \leq \frac{s^n}{r^n} = \left(\tfrac{s}{r}\right)^n\,,$$

and the series $\sum\left(\frac{s}{r}\right)^n$ converges (its geometric with positive ratio $\frac{s}{r} < 1$), and so the series $\sum\left|c_n(x-a)^n\right|$ converges too, by the Comparison Test.

8.4 Definition: The number R in the above theorem is called the **radius of convergence** of the power series, and the interval I is called the **interval of convergence** of the power series.

8.5 Example: Find the interval of convergence of the power series $\displaystyle\sum_{n\geq 1}\frac{(3-2x)^n}{\sqrt{n}}$.

Solution: First note that this is in fact a power series, since $\dfrac{(3-2x)^n}{\sqrt{n}}=\frac{(-2)^n}{\sqrt{n}}\left(x-\frac{3}{2}\right)^n$, and so $\displaystyle\sum_{n\geq 1}\frac{(3-2x)^n}{\sqrt{n}}=\sum_{n\geq 0}c_n(x-a)^n$, where $c_0=0$, $c_n=\frac{(-2)^n}{\sqrt{n}}$ for $n\geq 1$ and $a=\frac{3}{2}$. Now, let $a_n=\dfrac{(3-2x)^n}{\sqrt{n}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(3-2x)^{n+1}}{\sqrt{n+1}}\;\frac{\sqrt{n}}{(3-2x)^n}\right|=\sqrt{\tfrac{n}{n+1}}\;|3-2x|\longrightarrow|3-2x|\text{ as }n\to\infty.$$

By the Ratio Test, $\sum a_n$ converges when $|3-2x|<1$ and diverges when $|3-2x|>1$. Equivalently, it converges when $x\in(1,2)$ and diverges when $x\notin[1,2]$. When $x=1$ so $(3-2x)=1$, we have $\sum a_n=\sum\frac{1}{\sqrt{n}}$, which diverges (its a p-series), and when $x=2$ so $(3-2x)=-1$, we have $\sum a_n=\sum\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $I=(1,2\,]$.

8.6 Note: An argument similar to the one used in the above example, using the Ratio Test, can be used to show that if $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ exists (finite or infinite) then the radius of convergence of the power series $\sum c_n(x-a)^n$ is equal to

$$R=\frac{1}{\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|}=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|\,.$$

Indeed if we let $R=\displaystyle\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ and write $a_n=c_n(x-a)^n$ then we have

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\left|c_{n+1}(x-a)^{n+1}\right|}{\left|c_n(x-a)^n\right|}=\left|\frac{c_{n+1}}{c_n}\right|\,|x-a|\longrightarrow\frac{1}{R}\,|x-a|$$

and so by the Ratio Test, if $|x-a|<R$ then $\sum|a_n|$ converges while if $|x-a|>R$ then $|a_n|\to\infty$ so $\sum a_n$ diverges. Thus R must be equal to the radius of convergence.

Operations on Power Series

8.7 Theorem: *(Continuity of Power Series) Suppose that the power series* $\sum c_n(x-a)^n$ *converges in an interval* I*. Then the sum* $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ *is continuous in* I*.*

Proof: We omit the proof

8.8 Theorem: *(Addition and Subtraction of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in the interval* I*. Then* $\sum(a_n+b_n)(x-a)^n$ *and* $\sum(a_n-b_n)(x-a)^n$ *both converge in* I*, and for all* $x\in I$ *we have*

$$\left(\sum_{n=0}^{\infty} a_n(x-a)^n\right) \pm \left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n\,.$$

Proof: This follows from Linearity.

8.9 Theorem: *(Multiplication of Power Series) Suppose the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*. Let* $c_n=\sum_{k=0}^{n} a_k b_{n-k}$*. Then* $\sum c_n(x-a)^n$ *converges in* I *and for all* $x\in I$ *we have*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \left(\sum_{n=0}^{\infty} a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right)\,.$$

Proof: This follows from the Multiplication of Series Theorem, since the power series converge absolutely in I.

8.10 Theorem: *(Division of Power Series) Suppose that* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*, and that* $b_0\neq 0$*. Define* c_n *by*

$$c_0=\tfrac{a_0}{b_0}\ \text{, and for } n>0,\ c_n=\tfrac{a_n}{b_0}-\tfrac{b_n c_0}{b_0}-\tfrac{b_{n-1}c_1}{b_0}-\cdots-\tfrac{b_1 c_{n-1}}{b_0}\,.$$

Then there is an open interval J *with* $a\in J$ *such that* $\sum c_n(x-a)^n$ *converges in* J *and for all* $x\in J$*,*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \frac{\sum_{n=0}^{\infty} a_n(x-a)^n}{\sum_{n=0}^{\infty} b_n(x-a)^n}\,.$$

Proof: We omit the proof.

8.11 Theorem: *(Composition of Power Series) Let* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *in an open interval* I *with* $a \in I$*, and let* $g(y) = \sum_{m=0}^{\infty} b_m(y-b)^m$ *in an open interval* J *with* $b \in J$ *and with* $a_0 \in J$*. Let* K *be an open interval with* $a \in K$ *such that* $f(K) \subset J$*. For each* $m \geq 0$*, let* $c_{n,m}$ *be the coefficients, found by multiplying power series, such that* $\sum_{n=0}^{\infty} c_{n,m}(x-a)^n = b_n\left(\sum_{n=0}^{\infty} a_n(x-a)^n - b\right)^m$*. Then* $\sum_{m\geq 0} c_{n,m}$ *converges for all* $m \geq 0$*, and for all* $x \in K$*,* $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n$ *converges and*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n = g\big(f(x)\big)\,.$$

Proof: We omit the proof.

8.12 Theorem: *(Integration of Power Series) Supoose that* $\sum c_n(x-a)^n$ *converges in the interval* I*. Then for all* $x \in I$*, the sum* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is integrable on* $[a,x]$ *(or* $[x,a]$*) and*

$$\int_a^x \sum_{n=0}^{\infty} c_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\int_a^x c_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}\,.$$

Proof: We omit the proof

8.13 Theorem: *(Differentiation of Power Series) Suppose that* $\sum c_n(x-a)^n$ *converges in the open interval* I*. Then the sum* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is differentiable in* I *and*

$$f'(x) = \sum_{n=1}^{\infty} n\,c_n(x-a)^{n-1}\,.$$

Proof: We omit the proof

8.14 Example: Find a power series centred at 0 whose sum is $f(x) = \dfrac{1}{x^2+3x+2}$, and find its interval of convergence.

Solution: We have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{\frac{1}{2}}{1+\frac{x}{2}} \\ &= \sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} \tfrac{1}{2}\left(-\tfrac{x}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} \tfrac{(-1)^n}{2^{n+1}} x^n \\ &= \sum_{n=0}^{\infty} (-1)^n \left(1 - \tfrac{1}{2^{n+1}}\right) x^n \,. \end{aligned}$$

Since $\sum_{n=0}^{\infty} (-x)^n$ converges if and only if $|x| < 1$ and $\sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n$ converges when $|x| < 2$, it follows from Linearity the the sum of these two series converges if and only if $|x| < 1$.

8.15 Example: Find a power series centred at -4 whose sum is $f(x) = \dfrac{1}{x^2+3x+2}$, and find its interval of convergence.

Solution: We have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+4)-3} - \frac{1}{(x+4)-2} \\ &= \frac{-\frac{1}{3}}{1-\frac{x+4}{3}} + \frac{\frac{1}{2}}{1-\frac{x+4}{2}} = \sum_{n=0}^{\infty} -\tfrac{1}{3}\left(\tfrac{x+4}{3}\right)^n + \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{x+4}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\tfrac{1}{2^{n+1}} - \tfrac{1}{3^{n+1}}\right)(x+4)^n \,. \end{aligned}$$

Since $\sum_{n=0}^{\infty} -\frac{1}{3}\left(\frac{x+4}{3}\right)^n$ converges when $|x+4| < 3$ and $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+4}{2}\right)^n$ converges if and only if $|x+4| < 2$, it follows that their sum converges if and only if $|x+4| < 2$.

8.16 Example: Find a power series centred at 0 whose sum is $f(x) = \dfrac{1}{(1-x)^2}$.

Solution: We provide three solutions. For the first solution, we multiply two power series. For $|x| < 1$ we have

$$\begin{aligned} f(x) &= \frac{1}{1-x} \cdot \frac{1}{1-x} \\ &= \left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \\ &= 1 + (1+1)x + (1+1+1)x^2 + (1+1+1+1)x^3 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n \,. \end{aligned}$$

For the second solution, we note that $f(x) = \dfrac{1}{1-2x+x^2}$ and we use long division.

$$
\begin{array}{r|l}
 & \;1+2x+3x^2+4x^3+5x^4+\cdots \\
1-2x+x^2 & \;1+0x+0x^2+0x^3+0x^4-\cdots \\
 & \;\underline{1-2x+x^2} \\
 & \qquad 2x-x^2 \\
 & \qquad \underline{2x-4x^2+2x^3} \\
 & \qquad\qquad 3x^2-2x^3 \\
 & \qquad\qquad \underline{3x^2-6x^3+3x^4} \\
 & \qquad\qquad\qquad 4x^3-8x^4+\cdots \\
 & \qquad\qquad\qquad \underline{4x^3-8x^4+\cdots} \\
 & \qquad\qquad\qquad\qquad 5x^4+\cdots
\end{array}
$$

For the third solution, we note that $\displaystyle\int \frac{1}{(1-x)^2} = \frac{1}{1-x}$ and we use differentiation.

$$
\begin{aligned}
\frac{1}{1-x} &= 1+x^2+x^3+x^4+x^5+\cdots \\
\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\left(1+x+x^2+x^3+x^4+x^5+\cdots\right) \\
\frac{1}{(1-x)^2} &= 1+2x+3x^2+4x^3+5x^4+\cdots\,.
\end{aligned}
$$

8.17 Example: Find a power series centred at 0 whose sum is $\ln(1+x)$.

Solution: For $|x|<1$ we have

$$
\begin{aligned}
\frac{1}{1+x} &= 1-x+x^2-x^3+\cdots \\
\ln(1+x) &= \int 1-x+x^2-x^3++\cdots\; dx \\
&= c+x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots
\end{aligned}
$$

Putting in $x=0$ gives $0=c$, and so

$$\ln(1+x) = \sum_{n=1}^{\infty} \tfrac{(-1)^{n+1}}{n} x^n = x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots\,.$$

8.18 Example: Find a power series centred at 0 whose sum is $f(x)=\tan^{-1}x$.

Solution: For $|x|<1$ we have

$$
\begin{aligned}
\frac{1}{1+x^2} &= 1-x^2+x^4-x^6+\cdots \\
\tan^{-1}x &= \int 1-x^2+x^4-x^6+\cdots\; dx \\
&= c+x-\tfrac{1}{3}\,x^3+\tfrac{1}{5}\,x^5-\tfrac{1}{7}\,x^7+\cdots
\end{aligned}
$$

Putting in $x=0$ gives $0=c$, and so

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}\, x^{2n+1} = x-\tfrac{1}{3}\,x^3+\tfrac{1}{5}\,x^5-\tfrac{1}{7}\,x^7+\cdots$$

Taylor Series

8.19 Theorem: *Suppose that* $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *in an open interval* I *centred at* a. *Then* f *is infinitely differentiable at* a *and for all* $n \geq 0$ *we have*

$$c_n = \frac{f^{(n)}(a)}{n!}\,,$$

where $f^{(n)}(a)$ *denotes the* n^{th} *derivative of* f *at* a.

Proof: By repeated application of the Differentiation of Power Series Theorem, for all $x \in I$, we have

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1} \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1)\, c_n (x-a)^{n-2} \\ f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2)\, c_n (x-a)^{n-3}\,, \end{aligned}$$

and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (x-a)^{n-k}$$

and so $f(a) = c_0$, $f'(a) = c_1$, $f''(a) = 2 \cdot 1\, c_2$ and $f'''(a) = 3 \cdot 2 \cdot 1\, c_3$, and in general

$$f^{(n)}(a) = n!\; c_n$$

8.20 Definition: Given a function $f(x)$ whose derivatives of all order exist at $x = a$, we define the **Taylor series** of $f(x)$ centred at a to be the power series

$$T(x) = \sum_{n \geq 0} c_n (x-a)^n \quad \text{where } c_n = \frac{f^{(n)}(a)}{n!}$$

and we define the l^{th} **Taylor Polynomial** of $f(x)$ centred at a to be the l^{th} partial sum

$$T_l(x) = \sum_{n=0}^{l} c_n (x-a)^n \quad \text{where } c_n = \frac{f^{(n)}(a)}{n!}$$

8.21 Example: Find the Taylor series centred at 0 for $f(x) = e^x$.

Solution: We have $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = 1$ and $c_n = \frac{1}{n!}$ for all $n \geq 0$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\, x^n = 1 + x + \tfrac{1}{2!}x^2 = \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots\,.$$

8.22 Example: Find the Taylor series centred at 0 for $f(x) = \sin x$.

Solution: We have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x$ and so on, so that in general $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$. It follows that $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$, so we have $c_{2n} = 0$ and $c_{2n+1} = \frac{(-1)^n}{(2n+1)!}$. Thus

$$T(x) = \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots .$$

8.23 Example: Find the Taylor series centred at 0 for $f(x) = \cos x$.

Solution: We have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f''''(x) = \cos x$ and so on, so that in general $f^{(2n)}(x) = (-1)^n \cos x$ and $f^{(2n+1)}(x) = (-1)^{n+1} \sin x$. It follows that $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$, so we have $c_{2n} = \frac{(-1)^n}{(2n)!}$ and $c_{2n+1} = 0$. Thus

$$T(x) = \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n)!} x^{2n} = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{5!}x^6 + \cdots .$$

8.24 Example: Find the Taylor series centred at 0 for $f(x) = (1+x)^p$ where $p \in \mathbb{R}$.

Solution: $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, and in general

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n} \,,$$

so $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p-1)$, and in general $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$, and so we have $c_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \tbinom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

where we use the notation

$$\tbinom{p}{0} = 1 \text{ , and for } n \geq 1,\ \tbinom{p}{n} = \tfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

8.25 Theorem: *(Taylor) Let $f(x)$ be infinitely differentiable in an open interval I with $a \in I$. Let $T_l(x)$ be the l^{th} Taylor polynomial for $f(x)$ centred at a. Then for all $x \in I$ there exists a number c between a and x such that*

$$f(x) - T_l(x) = \frac{f^{(l+1)}(c)}{(l+1)!}(x-a)^{l+1}\,.$$

Proof: When $x = a$ both sides of the above equation are 0. Suppose that $x > a$ (the case that $x < a$ is similar). Since $f^{(l+1)}$ is differentiable and hence continuous, by the Extreme Value Theorem it attains its maximum and minimum values, say M and m. Since $m \leq f^{(l+1)}(t) \leq M$ for all $t \in I$, we have

$$\int_a^{t_1} m\,dt \leq \int_a^{t_1} f^{(l+1)}(t)\,dt \leq \int_a^{t_1} M\,dt$$

that is

$$m(t_1 - a) \leq f^{(l)}(t_1) - f^{(l)}(a) \leq M(t_1 - a)$$

for all $t_1 > a$ in I. Integrating each term with respect to t_1 from a to t_2, we get

$$\tfrac{1}{2}m(t_2-a)^2 \leq f^{(l-1)}(t_2) - f^{(l)}(a)(t_2-a) \leq \tfrac{1}{2}M(t_t-a)^2$$

for all $t_2 > a$ in I. Integrating with respect to t_2 from a to t_3 gives

$$\tfrac{1}{3!}m(t_3-a)^3 \leq f^{(l-2)}(t_3) - f^{(l-2)}(a) - \tfrac{1}{2}f^{(l)}(a)(t_3-a)^3 \leq \tfrac{1}{3!}M(t_3-a)^3$$

for all $t_3 > a$ in I. Repeating this procedure eventually gives

$$\tfrac{1}{(l+1)!}m(t_{l+1}-a)^{l+1} \leq f(t_{l+1}) - T_l(t_{l+1}) \leq \tfrac{1}{(l+1)!}M(t_{l+1}-a)^{l+1}$$

for all $t_{l+1} > a$ in I. In particular $\frac{1}{(l+1)!}m(x-a)^{l+1} \leq f(x) - T_l(x) \leq \frac{1}{(l+1)!}M(x-a)^{l+1}$, so

$$m \leq \big(f(x) - T_l(x)\big)\tfrac{(l+1)!}{(x-a)^{l+1}} \leq M\,.$$

By the Intermediate Value Theorem, there is a number $c \in [a, x]$ such that

$$f^{(l+1)}(c) = \big(f(x) - T_l(x)\big)\frac{(l+1)!}{(x-a)^{l+1}}$$

.

8.26 Theorem: *The functions e^x, $\sin x$, $\cos x$ and $(1+x)^p$ are all exactly equal to the sum of their Taylor series centred at 0 in the interval of convergence.*

Proof: First let $f(x)=e^x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T_l(x)=\dfrac{e^c x^{l+1}}{(l+1)!}$ for some c between 0 and x, and so

$$\big|f(x)-T_l(x)\big|\leq\frac{e^{|x|}|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, we have $\lim\limits_{l\to\infty}\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, so $\lim\limits_{l\to\infty}\big(f(x)-T_l(x)\big)=0$, and so $f(x)=\lim\limits_{l\to\infty}T_l(x)=T(x)$.

Now let $f(x)=\sin x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T(x)=\dfrac{f^{(l+1)}(c)\,x^{l+1}}{(l+1)!}$ for some c between 0 and x. Since $f^{(l+1)}(x)$ is one of the functions $\pm\sin x$ or $\pm\cos x$, we have $\big|f^{(l+1)}(c)\big|\leq 1$ for all c and so

$$\big|f(x)-T(x)\big|\leq\frac{|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, $\lim\limits_{l\to\infty}\dfrac{|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, and so we have and $f(x)=T(x)$ as above.

Let $f(x)=\cos x$. For all $x\in\mathbb{R}$ we have

$$\begin{aligned}f(x)&=\cos x=\tfrac{d}{dx}\sin x\\&=\tfrac{d}{dx}\left(x-\tfrac{1}{3!}x^3+\tfrac{1}{5!}x^5-\tfrac{1}{7!}x^7+\cdots\right)\\&=1-\tfrac{1}{2!}x^2+\tfrac{1}{4!}x^4-\tfrac{1}{6!}x^6+\cdots\end{aligned}$$

which is the sum of its Taylor series, centred at 0.

Finally, let $f(x)=(1+x)^p$. The Taylor series centred at 0 is

$$T(x)=1+px+\tfrac{p(p-1)}{2!}x^2+\tfrac{p(p-1)(p-2)}{3!}x^3+\tfrac{p(p-1)(p-2)(p-3)}{4!}x^4+\cdots$$

and it converges for $|x|<1$. Differentiating the power series gives

$$T'(x)=p+\tfrac{p(p-1)}{1!}x+\tfrac{p(p-1)(p-2)}{2!}x^2+\tfrac{p(p-1)(p-2)(p-3)}{3!}x^3+\cdots$$

and so

$$\begin{aligned}(1+x)T'(x)&=p+\left(p+\tfrac{p(p-1)}{1!}\right)x+\left(\tfrac{p(p-1)}{1!}+\tfrac{p(p-1)(p-2)}{2!}\right)x^2\\&\qquad+\left(\tfrac{p(p-1)(p-2)}{2!}-\tfrac{p(p-1)(p-2)(p-3)}{3!}\right)x^3+\cdots\\&=p+\tfrac{p\cdot p}{1!}x+\tfrac{p\cdot p(p-1)}{2!}x^2+\tfrac{p\cdot p(p-1)(p-2)}{3!}x^3+\cdots\\&=p\,T(x)\,.\end{aligned}$$

Thus we have $(1+x)T'(x)=pT(x)$ with $T(0)=1$. This DE is linear since we can write it as $T'(x)-\frac{p}{1+x}T(x)=0$. An integrating factor is $\lambda=e^{\int-\frac{p}{1+x}\,dx}=e^{-p\ln(1+x)}=(1+x)^{-p}$ and the solution is $T(x)=(1+x)^{-p}\displaystyle\int 0\,dx=b(1+x)^p$ for some constant b. Since $T(0)=1$ we have $b=1$ and so $T(x)=(1+x)^p=f(x)$.

Applications

8.27 Example: Let $f(x) = \sin\left(\frac{1}{2}x^2\right)$. Find the 10^{th} derivative $f^{(10)}(0)$.

Solution: We have

$$\begin{aligned} f(x) &= \sin\left(\tfrac{1}{2}\,x^2\right) \\ &= \left(\tfrac{1}{2}\,x^2\right) - \tfrac{1}{3!}\left(\tfrac{1}{2}\,x^2\right)^3 + \tfrac{1}{5!}\left(\tfrac{1}{2}\,x^2\right)^5 - \cdots \\ &= \tfrac{1}{2}\,x^2 - \tfrac{1}{2^3\,3!}\,x^6 + \tfrac{1}{2^5\,5!}\,x^{10} - \cdots \end{aligned}$$

We have $c_{10} = \frac{1}{2^5\,5!}$ and so $f^{(10)}(0) = 10!\,c_{10} = \frac{10!}{2^5\,5!} = \frac{10\cdot9\cdot8\cdot7\cdot6}{2^5} = 5\cdot9\cdot7\cdot3 = 945\,.$

8.28 Example: Find $\displaystyle\lim_{x\to0} \frac{e^{-2x^2} - \cos 2x}{\left(\tan^{-1}x - \ln(1+x)\right)^2}$

Solution: We have

$$\begin{aligned} \frac{e^{-2x^2} - \cos 2x}{\left(\tan^{-1}x - \ln(1+x)\right)^2} &= \frac{\left(1-(2x^2)+\frac{1}{2!}(2x^2)^2 - \cdots\right) - \left(1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \cdots\right)}{\left(\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\right) - \left(x - \frac{1}{2}x^3 + \frac{1}{3}x^3 - \cdots\right)\right)^2} \\ &= \frac{\left(1 - 2x^2 + 2x^4 - \cdots\right) - \left(1 - 2x^2 + \frac{2}{3}x^4 - \cdots\right)}{\left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \cdots\right)^2} \\ &= \frac{\frac{4}{3}x^4 + \cdots}{\frac{1}{4}x^4 + \cdots} = \frac{1}{3} + c_1x + \cdots \longrightarrow \frac{1}{3} \text{ as } x \to 0\,. \end{aligned}$$

8.29 Example: Approximate the value of $\frac{1}{\sqrt{e}}$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned} \tfrac{1}{\sqrt{e}} = e^{-1/2} &= 1 - \left(\tfrac{1}{2}\right) + \tfrac{1}{2!}\left(\tfrac{1}{2}\right)^2 - \tfrac{1}{3!}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4!}\left(\tfrac{1}{2}\right)^4 - \cdots \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{2^2\,2!} - \tfrac{1}{2^3\,3!} + \tfrac{1}{2^4\,4!} - \cdots \\ &\cong 1 - \tfrac{1}{2} + \tfrac{1}{2^2\,2!} - \tfrac{1}{2^3\,3!} = 1 - \tfrac{1}{2} + \tfrac{1}{8} - \tfrac{1}{48} = \tfrac{29}{48} \end{aligned}$$

with absolute error $E \le \frac{1}{2^4\,4!} = \frac{1}{384}$, by the Alternating Series Test.

8.30 Example: Approximate the value of $\sqrt{e}$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned} \sqrt{e} = e^{1/2} &= 1 + \left(\tfrac{1}{2}\right) + \tfrac{1}{2!}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{3!}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4!}\left(\tfrac{1}{2}\right)^4 + \tfrac{1}{5!}\left(\tfrac{1}{2}\right)^5 + \cdots \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2^2\,2!} + \tfrac{1}{2^3\,3!} + \tfrac{1}{2^4\,4!} + \tfrac{1}{2^5\,5!} + \cdots \\ &\cong 1 + \tfrac{1}{2} + \tfrac{1}{2^2\,2!} + \tfrac{1}{2^3\,3!} = 1 + \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{48} = \tfrac{79}{48} \end{aligned}$$

with absolute error

$$\begin{aligned} E &= \tfrac{1}{2^4\,4!} + \tfrac{1}{2^5\,5!} + \tfrac{1}{2^6\,6!} + \tfrac{1}{2^7\,7!} + \tfrac{1}{2^8\,8!} + \cdots \\ &= \tfrac{1}{2^4\,4!}\left(\tfrac{1}{2\cdot5} + \tfrac{1}{2^2\cdot6\cdot5} + \tfrac{1}{2^3\cdot7\cdot6\cdot5} + \tfrac{1}{2^4\cdot8\cdot7\cdot6\cdot5} + \cdots\right) \\ &\le \tfrac{1}{2^4\,4!}\left(\tfrac{1}{2\cdot5} + \tfrac{1}{2^25^2} + \tfrac{1}{2^35^3} + \tfrac{1}{2^45^4} + \cdots\right) \\ &= \tfrac{1}{384}\cdot\frac{1}{1-\frac{1}{10}} = \tfrac{1}{384}\cdot\tfrac{10}{9} = \tfrac{5}{1728} < \tfrac{1}{100}\,, \end{aligned}$$

where we used the Comparison Test and the formula for the sum of a geometric series.

8.31 Example: Approximate the value of $\ln 2$ so the error is at most $\frac{1}{50}$

Solution: We provide two solutions. For both solutions, we use the fundtion

$$f(x)=\ln(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots$$

For the first solution, we put in $x=1$ to get

$$\begin{aligned}\ln 2=f(1)&=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots\\ &\cong 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots+\tfrac{1}{49}\end{aligned}$$

with absolute error $E\leq\frac{1}{50}$ by the Alternating Series Test. It would be cumbersome to add up the 49 terms in the above alternating sum, so we provide a second solution in which we put in $x=-\frac{1}{2}$. We have

$$\begin{aligned}\ln 2=-\ln\tfrac{1}{2}=-f\big(-\tfrac{1}{2}\big)&=-\left(-\tfrac{1}{2}\right)+\tfrac{1}{2}\left(-\tfrac{1}{2}\right)^2-\tfrac{1}{3}\left(-\tfrac{1}{2}\right)^3+\tfrac{1}{4}\left(-\tfrac{1}{2}\right)^4-\cdots\\ &=\tfrac{1}{2}+\tfrac{1}{2\cdot 2^2}+\tfrac{1}{3\cdot 2^3}+\tfrac{1}{4\cdot 2^4}+\tfrac{1}{5\cdot 2^5}+\tfrac{1}{6\cdot 2^6}+\cdots\\ &\cong\tfrac{1}{2}+\tfrac{1}{2\cdot 2^2}+\tfrac{1}{3\cdot 2^3}+\tfrac{1}{4\cdot 2^4}=\tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{24}+\tfrac{1}{64}=\tfrac{131}{192}\end{aligned}$$

with absolute error

$$\begin{aligned}E&=\tfrac{1}{5\cdot 2^5}+\tfrac{1}{6\cdot 2^6}+\tfrac{1}{7\cdot 2^7}+\tfrac{1}{8\cdot 2^8}+\cdots\\ &\leq\tfrac{1}{5\cdot 2^5}+\tfrac{1}{5\cdot 2^6}+\tfrac{1}{5\cdot 2^7}+\tfrac{1}{5\cdot 2^8}+\cdots\\ &=\frac{\frac{1}{5\cdot 2^5}}{1-\frac{1}{2}}=\tfrac{2}{5\cdot 2^5}=\tfrac{1}{80}\end{aligned}$$

by the Comparison Test and the formula for the sum of a geometric series.

8.32 Example: Approximate the value of $10^{2/3}$ so the error is at most $\frac{1}{100}$.

Solution: We use the function

$$f(x)=(1+x)^{2/3}=1+\frac{\left(\frac{2}{3}\right)}{1!}\,x\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\,x^2+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}\,x^3+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{4!}\,x^4+\cdots$$

We have

$$\begin{aligned}10^{2/3}&=(8+2)^{2/3}=4\big(1+\tfrac{1}{4}\big)^{2/3}=4\,f\left(\tfrac{1}{4}\right)\\ &=4\left(1+\frac{\left(\frac{2}{3}\right)}{4\cdot 1!}+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{4^2\cdot 2!}+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{4^3\cdot 3!}+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{4^4\cdot 4!}+\cdots\right)\\ &=4+\tfrac{8}{12\cdot 1!}-\tfrac{8\cdot 1}{12^2\cdot 2!}+\tfrac{8\cdot 1\cdot 4}{12^3\cdot 3!}-\tfrac{8\cdot 1\cdot 4\cdot 7}{12^4\cdot 4!}+\cdots\\ &\cong 4+\tfrac{8}{12\cdot 1!}-\tfrac{8\cdot 1}{12^2\cdot 2!}=4+\tfrac{2}{3}-\tfrac{1}{36}=\tfrac{167}{36}\end{aligned}$$

with absolute error $E\leq\frac{8\cdot 1\cdot 4}{12^3\cdot 3!}=\frac{1}{324}$ by the Alternating Series Test.

8.33 Example: Approximate the value of π so the error is at most $\frac{1}{50}$.

Solution: We provide two solutions. For both solutions we use the function

$$f(x) = \tan^{-1} x = x - \tfrac{1}{3}\,x^3 + \tfrac{1}{5}\,x^5 - \tfrac{1}{7}\,x^7 + \cdots$$

For the first solution, we put in $x = 1$ to get

$$\pi = 4 \cdot \tfrac{\pi}{4} = 4f(1) = 4\big(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots\big) \cong 4\big(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots + \tfrac{1}{399}\big)$$

with absolute error $E \leq \frac{4}{201}$ by the Alternating Series Test. It would be cumbersome to add up the 100 terms in the alternating sum, so we provide a second solution in which we put in $x = \frac{1}{\sqrt{3}}$. We have

$$\begin{aligned}
\pi = 6 \cdot \tfrac{\pi}{6} = 6f\big(\tfrac{1}{\sqrt{3}}\big) &= 6\Big(\tfrac{1}{\sqrt{3}} - \tfrac{1}{3\cdot\sqrt{3}^{\,3}} + \tfrac{1}{5\cdot\sqrt{3}^{\,5}} - \tfrac{1}{7\cdot\sqrt{3}^{\,7}} + \tfrac{1}{9\cdot\sqrt{3}^{\,9}} - \cdots\Big) \\
&= 2\sqrt{3}\left(1 - \tfrac{1}{3\cdot 3} + \tfrac{1}{5\cdot 3^2} - \tfrac{1}{7\cdot 3^3} + \tfrac{1}{9\cdot 3^4} - \cdots\right) \\
&\cong 2\sqrt{3}\left(1 - \tfrac{1}{3\cdot 3} + \tfrac{1}{5\cdot 3^2}\right) = \tfrac{82\sqrt{3}}{45}
\end{aligned}$$

with absolute error $E \leq \frac{2\sqrt{3}}{7\cdot 3^3} = \frac{2\sqrt{3}}{189}$ by the Alternating Series Test. We remark that in order to make this approximation, we must first approximate $\sqrt{3}$.

8.34 Example: Approximate the value of $\sin\big(10°\big)$ so the error is at most $\frac{1}{1000}$.

Solution: We use the function

$$f(x) = \sin x = x - \tfrac{1}{3!}\,x^3 + \tfrac{1}{5!}\,x^5 - \cdots$$

We put in $x = 10° = \frac{\pi}{18}$ to get

$$\sin(10°) = f\left(\tfrac{\pi}{18}\right) = \tfrac{\pi}{18} - \tfrac{1}{3!}\left(\tfrac{\pi}{18}\right)^3 + \tfrac{1}{5!}\left(\tfrac{\pi}{18}\right)^5 - \cdots \cong \tfrac{\pi}{18}$$

with absolute error $E \leq \frac{1}{3!}\left(\frac{\pi}{18}\right)^3$ by the Alternating Series Test. We remark that in order to make this approximation, we must first approximate π.

8.35 Example: Approximate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned}
\int_0^1 e^{-x^2}\,dx &= \int_0^1 \Big(1 - x^2 + \tfrac{1}{2!}\,x^4 - \tfrac{1}{3!}\,x^6 + \tfrac{1}{4!}\,x^8 - \cdots\Big)\,dx \\
&= \Big[x - \tfrac{1}{3}\,x^3 + \tfrac{1}{5\cdot 2!}\,x^5 - \tfrac{1}{7\cdot 3!}\,x^7 + \tfrac{1}{9\cdot 4!}\,x^9 - \cdots\Big]_0^1 \\
&= 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2!} - \tfrac{1}{7\cdot 3!} + \tfrac{1}{9\cdot 4!} - \cdots \\
&\cong 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2!} - \tfrac{1}{7\cdot 3!} = \tfrac{26}{35}
\end{aligned}$$

with absolute error $E \leq \frac{1}{9\cdot 4!} = \frac{1}{216}$ by the Alternating Series Test.

8.36 Example: Approximate the value of $\displaystyle\int_0^{\sqrt{2}} \frac{\sin x}{x}\,dx$ so the error is at most $\frac{1}{50}$.

8.37 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!}$.

Solution: We have

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{2}^{\,2n}}{(2n)!} = \cos(\sqrt{2})\,.$$

8.38 Example: Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{(-3)^n}$.

Solution: Note first that

$$\sum_{n=1}^{\infty} \frac{n-2}{(-3)^n} = \sum_{n=1}^{\infty} \frac{n}{(-3)^n} - \sum_{n=1}^{\infty} \frac{2}{(-3)^n}\,.$$

The second sum on the right is geometric with first term $-\frac{2}{3}$ and ratio $-\frac{1}{3}$, so we have

$$\sum_{n=1} \frac{2}{(-3)^n} = \frac{-\frac{2}{3}}{1+\frac{1}{3}} = -\tfrac{1}{2}\,.$$

To find the first sum on the right, we begin with the fact that for $|x| < 1$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Differentiate both sides to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^3 + 4x^3 + \cdots$$

Multiply both sides by x to get

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

Thus we obtain the formula

$$\sum_{n=1}^{\infty} n\,x^n = \frac{x}{(1-x)^2} \text{ for all } |x| < 1\,.$$

Put in $x = -\frac{1}{3}$ to get

$$\sum_{n=1}^{\infty} \frac{n}{(-3)^n} = \frac{-\frac{1}{3}}{\left(1+\frac{1}{3}\right)^2} = -\tfrac{3}{16}\,.$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{n}{(-3)^n} = \sum_{n=1}^{\infty} \frac{n}{(-3)^n} - \sum_{n=1}^{\infty} \frac{2}{(-3)^n} = -\tfrac{3}{16} + \tfrac{1}{2} = \tfrac{5}{16}\,.$$

8.39 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty} \frac{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)}{5^n\, n!}$.

Solution: We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)}{5^n\, n!} &= 2\sum_{n=0}^{\infty} \frac{\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\cdots\left(\frac{3n+2}{3}\right)}{n!}\cdot\frac{3^n}{5^n} \\
&= 2\sum_{n=0}^{\infty} \frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)\cdots\left(-\frac{3n+2}{3}\right)}{n!}\cdot\left(-\tfrac{3}{5}\right)^n \\
&= 2\left(1-\tfrac{3}{5}\right)^{-5/3} = 2\cdot\left(\tfrac{5}{2}\right)^{5/3}\ .
\end{aligned}$$