MATH 138 Calculus 2, Solutions to the Exercises for Chapter 1

1: (a) Find $\displaystyle\int_1^2 2x^3 - 3x^2 + x - 4\ dx$.

Solution: $\displaystyle\int_1^2 2x^3 - 3x^2 + x - 4\ dx = \Big[\tfrac{1}{2}\,x^4 - x^3 + \tfrac{1}{2}\,x^2 - 4x\Big]_1^2 = \big(8 - 8 + 2 - 8\big) - \big(\tfrac{1}{2} - 1 + \tfrac{1}{2} - 4\big) = -2.$

(b) Find $\displaystyle\int_0^4 \sqrt{2x+1}\ dx$.

Solution: $\displaystyle\int_0^4 \sqrt{2x+1}\ dx = \Big[\tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4 = \tfrac{1}{3}(27-1) = \tfrac{26}{3}.$

(c) Find $\displaystyle\int_1^4 \frac{2x^2 + \sqrt{x} + 1}{x}\ dx$.

Solution: $\displaystyle\int_1^4 \frac{2x^2 + \sqrt{x} + 1}{x}\ dx = \int_1^4 2x + x^{-1/2} + \frac{1}{x}\ dx = \Big[x^2 + 2x^{1/2} + \ln x\Big]_1^4 = (16+4+\ln 4) - (1+2) - 17 + \ln 4.$

(d) Find $\displaystyle\int_0^{2\pi} \big|\cos x - \sqrt{3}\sin x\big| dx$.

Solution: Let $f(x) = \cos x - \sqrt{3}\sin x$. We must determine where $f(x)$ is positive. One way to do this is to sketch $y = \cos x$ and $y = \sqrt{3}\sin x$ on the same grid. Another is as follows. Note that for $0 \le x \le 2\pi$ we have

$$f(x) = 0 \iff \sqrt{3}\sin x = \cos x \iff \tan x = \tfrac{1}{\sqrt{3}} \iff x = \tfrac{\pi}{6}, \tfrac{7\pi}{6}$$

and $f(0) = 1 > 0$, $f(\pi) = -1 < 0$ and $f(2\pi) = 1 > 0$, and so $f(x) \ge 0$ for $x \in \big[0, \tfrac{\pi}{6}\big] \cup \big[\tfrac{7\pi}{6}, 2\pi\big]$ and $f(x) \le 0$ for $x \in \big[\tfrac{\pi}{6}, \tfrac{7\pi}{6}\big]$. Thus

$$\begin{aligned}\int_0^{2\pi} \big|f(x)\big| dx &= \int_0^{\pi/6} f(x)\ dx - \int_{\pi/6}^{7\pi/6} f(x)\ dx + \int_{7\pi/6}^{2\pi} f(x)\ dx \\ &= \Big[\sin x + \sqrt{3}\cos x\Big]_0^{\pi/6} - \Big[\sin x + \sqrt{3}\cos x\Big]_{\pi/6}^{7\pi/6} + \Big[\sin x + \sqrt{3}\cos x\Big]_{7\pi/6}^{2\pi} \\ &= \big(\tfrac{1}{2} + \tfrac{3}{2}\big) - \big(0 + \sqrt{3}\big) - \big(\big(-\tfrac{1}{2} - \tfrac{3}{2}\big) - \big(\tfrac{1}{2} + \tfrac{3}{2}\big)\big) + \big(0 + \sqrt{3}\big) - \big(-\tfrac{1}{2} - \tfrac{3}{2}\big) = 8\,.\end{aligned}$$

We remark that there is an alternate solution which makes use of the fact that $f(x) = -2\sin\big(x - \tfrac{\pi}{6}\big)$.

2: (a) Approximate $\int_0^{2\pi} 4^{\cos x}\,dx$ by the Riemann Sum for $f(x)=4^{\cos x}$ using the right endpoints of 6 equal-sized subintervals of $[0,2\pi]$.

Solution: The subintervals all have size $\frac{\pi}{3}$ and the endpoints of the subintervals are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$, so $\int_0^{2\pi} f(x)\,dx \cong \frac{\pi}{3}\Big(f\left(\frac{\pi}{3}\right)+f\left(\frac{2\pi}{3}\right)+f(\pi)+f(\frac{4\pi}{3})+f(\frac{5\pi}{3})+f(2\pi)\Big) = \frac{\pi}{3}\left(4^{1/2}+4^{-1/2}+4^{-1}+4^{-1/2}+4^{1/2}+4^{1}\right)$ $= \frac{\pi}{3}\left(2+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+2+4\right) = \frac{37\pi}{12}$

(b) Find the exact value of $\int_{-1}^{2} x^2+2x\;dx$ by calculating the limit of a sequence of Riemann sums.

Solution: Let $f(x)=x^2+2x$. Then

$$\begin{aligned}
\int_{-1}^{2} f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)(x_i-x_{i-1}) = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\tfrac{3}{n}\,i\right)\left(\tfrac{3}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\Big(\left(-1+\tfrac{3}{n}\,i\right)^2+2\left(-1+\tfrac{3}{n}\,i\right)\Big)\left(\tfrac{3}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1-\tfrac{6}{n}\,i+\tfrac{9}{n^2}\,i^2-2+\tfrac{6}{n}\,i\right)\left(\tfrac{3}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(-1+\tfrac{9}{n^2}\,i^2\right)\left(\tfrac{3}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(-\tfrac{3}{n}+\tfrac{27}{n^3}\,i^2\right) = \lim_{n\to\infty}\left(-\tfrac{3}{n}\sum_{i=1}^{n}1+\tfrac{27}{n^3}\sum_{i=1}^{n}i^2\right)\\
&= \lim_{n\to\infty}\left(-\tfrac{3}{n}\cdot n+\tfrac{27}{n^3}\cdot\tfrac{n\,(n+1)(2n+1)}{6}\right) = -3+9=6\,.
\end{aligned}$$

3: (a) Let $h(x)=\int_0^x \sqrt{1+t^3}\,dt$. Find $h'(2)$.

Solution: By the FTC, $h'(x)=\sqrt{1+x^3}$ so $h'(2)=\sqrt{1+8}=3$.

(b) Let $h(x)=\int_{\sqrt{x}}^{x\sqrt{x}} 2^{3-\sqrt{t/2}}\,dt$. Find $h'(4)$.

Solution: Let $f(t)=2^{3-\sqrt{t/2}}$. Let $g(u)=\int_0^u f(t)\,dt$, and note that $g'(u)=f(u)$ by the FTC. Let $u(x)=\sqrt{x}$ and $v(x)=x\sqrt{x}$ so that $h(x)=g(v(x))-g(u(x))$, and note that $u'(x)=\frac{1}{2\sqrt{x}}$ and $v'(x)=\frac{3}{2}\sqrt{x}$. Then $h'(x)=g'(v(x))v'(x)-g'(u(x))u'(x)=f(v(x))v'(x)-f(u(x))u'(x)$, and so in particular $h'(4)=f(v(4))v'(4)-f(u(4))u'(4)=f(8)v'(4)-f(2)u'(4)=2\cdot 3-4\cdot\frac{1}{4}=6-1=5$.