MATH 138 Calculus 2, Solutions to the Exercises for Chapter 2

1: Evaluate each of the following definite integrals.

(a) $\int_1^3 x\,\sqrt[3]{x^2-1}\,dx$

Solution: We use the substitution $u=x^2-1$ so that $du=2x\,dx$ to get

$$\int_1^3 x\,(x^2-1)^{\frac{1}{3}}\,dx=\int_0^8 \tfrac{1}{2}\,u^{\frac{1}{3}}\,du=\Big[\tfrac{3}{8}\,u^{4/3}\Big]_0^8=\tfrac{3}{8}(16-0)=6\,.$$

(b) $\int_0^{\pi/2}\frac{\cos^3 x}{1+\sin^2 x}\,dx$

Solution: We use the substitution $u=\sin x$ so $du=\cos x\,dx$ to get

$$\int_0^{\pi/2}\frac{\cos^3 x}{1+\sin^2 x}\,dx=\int_0^{\pi/2}\frac{(1-\sin^2 x)\cos x}{1+\sin^2 x}\,dx=\int_0^1\frac{1-u^2}{1+u^2}\,du=\int_0^1 -1+\frac{2}{1+u^2}\,du$$
$$=\Big[-u+2\tan^{-1}u\Big]_0^1=\big(-1+\tfrac{\pi}{2}\big)-(0)=\tfrac{\pi}{2}-1\,.$$

2: Evaluate each of the following definite integrals.

(a) $\int_1^4\frac{\ln x}{\sqrt{x}}\,dx$

Solution: We integrate by parts using $u=\ln x$, $du=\frac{1}{x}\,dx$, $v=2x^{1/2}$ and $dv=x^{-1/2}\,dx$ to get

$$\int_1^4\frac{\ln x}{\sqrt{x}}\,dx=\Big[2x^{1/2}\ln x-\int 2x^{-1/2}\,dx\Big]_1^4=\Big[2x^{1/2}\ln x-4x^{1/2}\Big]_1^4=(4\ln 4-8)-(-4)=4\ln 4-4\,.$$

(b) $\int_0^2 x^2e^{x/2}\,dx$

Solution: We integrate by parts twice, the first time using $u_1=x^2$, $du_1=2x\,dx$, $v_1=2e^{x/2}$, $dv_1=e^{x/2}\,dx$, and the second time using $u_2=4x$, $du_2=4\,dx$, $v_2=v_1=2e^{x/2}$, $dv_2=dv_1=e^{x/2}\,dx$, to get

$$\int_0^2 x^2e^{x/2}\,dx=\Big[2x^2e^{x/2}-\int 4x\,e^{x/2}\,dx\Big]_0^2=\Big[2x^2e^{x/2}-\Big(8x\,e^{x/2}-\int 8\,e^{x/2}\,dx\Big)\Big]_0^2$$
$$=\Big[2x^2e^{x/2}-8x\,e^{x/2}+16\,e^{x/2}\Big]_0^2=\Big[2(x^2-4x+8)\,e^{x/2}\Big]_0^2=2(4e-8)=8e-16\,.$$

3: Find the following indefinite integrals.

(a) $\displaystyle\int (x^2+1)\,e^x\;dx$

Solution: Let $u = x^2+1$ and $v = e^x$. Then $\displaystyle\int (x^2+1)\,e^x\;dx = \int u\,dv = uv - \int v\,du = (x^2+1)\,e^x - \int 2x\,e^x\;dx.$ Let $u_1 = 2x$ and $v_1 = e^x$. Then $\displaystyle\int 2x\,e^x\;dx = \int u_1\,dv_1 = u_1v_1 - \int v_1\,du_1 = 2x\,e^x - \int 2\,e^x = 2x\,e^x - 2\,e^x + a.$ Thus $\displaystyle\int (x^2+1)\,e^x\;dx = (x^2+1)\,e^x - \int 2x\,e^x\;dx = (x^2+1)\,e^x - 2x\,e^x + 2\,e^x - a = (x^2-x+3)\,e^x + c$

(b) $\displaystyle\int \sin^3 x\cos^2 x\;dx$

Solution: Let $u = \cos x$ so $du = -\sin x\;dx$. Then $\displaystyle\int \sin^3 x\cos^2 x\;dx = \int (1-\cos^2 x)\cos^2 x\;\sin x\,dx = \int (u^2-1)u^2\;du = \int u^4 - u^2\;du = \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + c = \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + c.$

4: Evaluate each of the following definite integrals.

(a) $\displaystyle\int_0^{\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+1}}$

Solution: We use the substitution $u = \sqrt{x^2+1}$ so $du = \frac{x}{\sqrt{x^2+1}}\,dx$ and $x^2 = u^2-1$ to get

$$\int_0^{\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+1}} = \int_1^2 u^2-1\;du = \Big[\tfrac{1}{3}\,u^3 - u\Big]_1^2 = \big(\tfrac{8}{3}-2\big) - \big(\tfrac{1}{3}-1\big) = \tfrac{4}{3}\,.$$

(b) $\displaystyle\int_0^1 x\,\tan^{-1} x\;dx$

Solution: We integrate by parts using $u = \tan^{-1} x$, $du = \frac{1}{1+x^2}\,dx$, $v = \frac{1}{2}\,x^2\,dx$ and $dv = x\,dx$ to get

$$\int_0^1 x\tan^{-1} x\;dx = \Big[\tfrac{1}{2}\,x^2\tan^{-1} x - \int \tfrac{1}{2}\,\tfrac{x^2}{1+x^2}\,dx\Big]_0^1 = \Big[\tfrac{1}{2}\,x^2\tan^{-1} x - \tfrac{1}{2}\int 1 - \tfrac{1}{1+x^2}\;dx\Big]_0^1$$
$$= \Big[\tfrac{1}{2}\,x^2\tan^{-1} x + \tfrac{1}{2}\tan^{-1} x - \tfrac{1}{2}\,x\Big]_0^1 = \big(\tfrac{\pi}{8}+\tfrac{\pi}{8}-\tfrac{1}{2}\big) - (0) = \tfrac{\pi}{4} - \tfrac{1}{2}\,.$$

5: Evaluate the following definite integrals.

(a) $\displaystyle\int_0^4 \frac{x+2}{\sqrt{2x+1}}\;dx$

Solution: Let $u = 2x+1$ so $x = \frac{u-1}{2}$ and $du = 2\,dx$. Then $\displaystyle\int_0^4 \frac{x+2}{\sqrt{2x+1}}\;dx = \int_1^9 \frac{\frac{u-1}{2}+2}{\sqrt{u}}\,\tfrac{1}{2}du = \tfrac{1}{4}\int_1^9 \frac{u+3}{\sqrt{u}}\;du = \tfrac{1}{4}\int_1^9 u^{1/2} + 3u^{-1/2}\;du = \tfrac{1}{4}\Big[\tfrac{2}{3}u^{3/2} + 6u^{1/2}\Big]_1^9 = \tfrac{1}{4}\Big[(18+18) - \big(\tfrac{2}{3}+6\big)\Big] = \tfrac{22}{3}.$

(b) $\displaystyle\int_0^{\pi/6} \sec^4 2x\;dx$

Solution: Let $u = \tan 2x$ so $du = 2\sec^2 2x\,dx$. Then $\displaystyle\int_0^{\pi/6} \sec^4 2x\;dx = \int_0^{\pi/6} (1+\tan^2 2x)\;\sec^2 2x\,dx = \tfrac{1}{2}\int_0^{\sqrt{3}} (1+u^2)\,du = \tfrac{1}{2}\Big[u + \tfrac{1}{3}u^3\Big]_0^{\sqrt{3}} = \sqrt{3}.$

6: Find the following indefinite integrals.

(a) $\displaystyle\int \frac{4x}{(x+1)(x^2+4x+5)}\,dx$

Solution: Use partial fractions. To get $\dfrac{4x}{(x+1)(x^2+4x+5)} = \dfrac{A}{x+1} + \dfrac{B(2x+4)+C}{x^2+4x+5}$, we need to have $A(x^2+4x+5)+B(2x+4)(x+1)+C(x+1) = 4x$. We equate coeffitients to get $A+2B=0$, $4A+6B+C=4$ and $5A+4B+C=0$. Solve these three equations to get $A=-2$, $B=1$ and $C=6$. So $\displaystyle\int \frac{4x}{(x+1)(x^2+4x+5)}\,dx = \int \frac{-2}{x+1} + \frac{2x+4}{x^2+4x+5} + \frac{6}{(x+2)^2+1}\,dx = -2\ln|x+1| + \ln(x^2+4x+5) + 6\tan^{-1}(x+2)+c$.

(b) $\displaystyle\int \frac{x^2}{\sqrt{1-4x^2}}\,dx$

Solution: Let $\sin\theta = 2x$, so $\cos\theta = \sqrt{1-4x^2}$ and $\cos\theta\,d\theta = 2\,dx$. Then $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-4x^2}} = \int \frac{\frac{1}{4}\sin^2\theta\frac{1}{2}\cos\theta\,d\theta}{\cos\theta}$ $\displaystyle = \tfrac{1}{8}\int \sin^2\theta\;d\theta = \tfrac{1}{16}\int (1-\cos 2\theta)\;d\theta = \tfrac{1}{16}\big(\theta - \tfrac{1}{2}\sin 2\theta\big)+c = \tfrac{1}{16}\big(\theta - \sin\theta\cos\theta\big)+c = \tfrac{1}{16}\sin^{-1}2x - \tfrac{1}{8}x\sqrt{1-4x^2}+c$.

7: Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 (3+2x-x^2)^{-3/2}\,dx$

Solution: Let $2\sin\theta = x-1$ so $2\cos\theta = \sqrt{4-(x-1)^2} = \sqrt{3+2x-x^2}$ and $2\cos\theta\,d\theta = dx$. Then $\displaystyle\int_0^1 \frac{dx}{(3+2x-x^2)^{3/2}} = \int_{x=0}^1 \frac{2\cos\theta\,d\theta}{(2\cos\theta)^3} = \frac{1}{4}\int_{x=0}^1 \sec^2\theta\;d\theta = \frac{1}{4}\Big[\tan\theta\Big]_{x=0}^1 = \frac{1}{4}\left[\frac{x-1}{\sqrt{3+2x-x^2}}\right]_0^1 = \frac{1}{4\sqrt{3}}$.

(b) $\displaystyle\int_1^2 \frac{5x^2+9}{x^4-9x^2}\,dx$

Solution: To get $\dfrac{5x^2+9}{x^4-9x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-3} + \dfrac{D}{x+3}$, we need $A(x^3-9x)+B(x^2-9)+C(x^3+3x^2)+D(x^3-3x) = 5x^2+9$. Equate coefficients to get $A+C+D=0$, $B+3C-3D=5$, $-9A=0$ and $-9B=0$. Solve these 4 equations to get $A=0$, $B=-1$, $C=1$ and $D=-1$. So $\displaystyle\int_1^2 \frac{5x^2+9}{x^4-9x^2}\,dx = \int_1^2 -\frac{1}{x^2} + \frac{1}{x-3} - \frac{1}{x+3}\,dx = \Big[\tfrac{1}{x} + \ln(x-3) - \ln|x+3|\Big]_1^2 = \big(\tfrac{1}{2}+\ln 1 - \ln 5\big) - \big(1+\ln 2 - \ln 4\big) = -\tfrac{1}{2} - \ln\tfrac{5}{2}$