MATH 138 Calculus 2, Solutions to the Exercises for Chapter 3

1: A tank initially contains 500 L of water. Water drains out at a rate $r(t)$ L/min. Some values of $r(t)$ are tabulated below.

t	0	10	20	30	40	50	60
r	1.0	1.4	2.0	2.8	4.0	5.8	8.0

(a) Estimate the amount of water remaining in the tank after one hour by approximating the definite integral $\int_0^{60} r(t)\,dt$ using the midpoint rule on 3 subintervals.

Solution: The amount of water that drained out is

$$\int_0^{60} r(t)\,dt \cong \big(r(10)+r(30)+r(50)\big)(20) = (1.4+2.8+5.8)(20) = (10)(20) = 200\,.$$

The number of liters remaining in the tank is approximately $300 - 200 = 300$.

(b) Estimate the amount of water remaining in the tank after one hour by approximating the same integral using the Trapezoidal Rule on 6 subintervals.

Solution: Using the Trapezoid Rule T_6, the amount of water that drained out is

$$\begin{aligned}\int_0^{60} r(t)\,dt &\cong \tfrac{1}{2}\big(r(10)+2r(20)+2r(30)+2r(40)+2r(50)+r(60)\big)(10)\\ &= \tfrac{1}{2}\,(1.0+2.8+4.0+5.6+8.0+11.6+8.0)(10)\\ &= \tfrac{1}{2}(41)(10) = 205\end{aligned}$$

so the number of litres remaining is approximately $300 - 205 = 295$.

2: Suppose that $\left|f''(x)\right| \le \frac{1}{2}$ and $\left|f''''(x)\right| \le 2$ for all $x \in [0,24]$, and that $f(x)$ has the following table of values.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$f(x)$	4	5	4	3	2	2	2	3	4	5	6	3	2	1	1	2	4	4	4	3	71	1	3	4	7

(a) Use S_8 to approximate $\int_0^{24} f(x)\,dx$.

Solution: We have

$$\begin{aligned}\int_0^{24} f(x)\,dx &\cong \frac{24}{3\cdot 8}\cdot\big(f(0)+4f(3)+2f(6)+4f(9)+2f(12)+4f(15)+2f(18)+4f(21)+2f(24)\big)\\ &= 1\cdot\big(4+12+4+20+4+8+8+4+7\big) = 71.\end{aligned}$$

(b) Find a value of n such that if we estimate $\int_0^{24} f(x)\,dx$ using T_n the error is $E \le .01$.

Solution: The error is $E \le \dfrac{(b-a)^3}{12n^2}\max\limits_{0\le x\le 24}\left|f''(x)\right| = \dfrac{24^3}{12n^2}\,\dfrac{1}{2} = \dfrac{24^2}{n^2}$. To get $E \le \frac{1}{100}$ we need $\left(\frac{24}{n}\right)^2 \le \frac{1}{100}$, that is $\frac{24}{n} \le \frac{1}{10}$, so we can take $n \ge 240$.

3: (a) Approximate $\displaystyle\int_0^{2\pi} 4^{\cos x}\,dx$ using R_6.

Solution: The subintervals all have size $\frac{\pi}{3}$ and the endpoints of the subintervals are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$, so $\displaystyle\int_0^{2\pi} f(x)\,dx \cong \frac{\pi}{3}\Big(f\left(\frac{\pi}{3}\right)+f\left(\frac{2\pi}{3}\right)+f(\pi)+f\left(\frac{4\pi}{3}\right)+f\left(\frac{5\pi}{3}\right)+f(2\pi)\Big) = \frac{\pi}{3}\left(4^{1/2}+4^{-1/2}+4^{-1}+4^{-1/2}+4^{1/2}+4^{1}\right)$ $= \frac{\pi}{3}\left(2+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+2+4\right) = \frac{37\pi}{12}$.

(b) Find a value of n such that if we approximate $\displaystyle\int_0^4 3\sqrt{2x+1}\,dx$ using S_n then the error is $E \leq .0001$.

Solution: Let $f(x) = 3\sqrt{2x+1}$. Then $f'(x) = 3(2x+1)^{-1/2}$, $f''(x) = -3(2x+1)^{-3/2}$, $f''' = 9(2x+1)^{-5/2}$ and $f''''(x) = -45(2x+1)^{-7/2}$, so for $0 \leq x \leq 4$ we have $|f''''(x)| \leq 45$. If E_n denotes the error, then $E_n = \dfrac{(b-a)^5}{180n^4}\max|f''''(x)| = \dfrac{4^5}{180n^4}\,45 = \dfrac{4^4}{n^4}$. To get $\frac{4^4}{n^4} \leq 10^{-4}$ we need $n^4 \geq 4^4\cdot 10^4$, so we take $n = 40$.

4: Evaluate the following improper integrals.

(a) $\displaystyle\int_0^2 x^3\ln(x/2)\,dx$

Solution: Let $u = \ln(x/2)$ so $du = \frac{1}{x}\,dx$, and let $v = \frac{1}{4}x^4$ so $dv = x^3\,dx$. Then $\displaystyle\int_0^2 x^3\ln(x/2)\,dx = \int_0^2 u\,dv =$ $\displaystyle\Big[uv - \int v\,du\Big]_0^2 = \Big[\tfrac{1}{4}x^4\ln(x/2) - \int \tfrac{1}{4}x^3\,dx\Big]_{0^+}^2 = \Big[\tfrac{1}{4}x^4\ln(x/2) - \tfrac{1}{16}x^4\Big]_{0^+}^2 = -1$, since $\displaystyle\lim_{x\to 0^+} x^4\ln(x/2) = 0$.

(b) $F(s) = \displaystyle\int_0^\infty e^{-st}\sin t\,dt$, where $s > 0$.

Solution: Let $I(s) = \displaystyle\int e^{-st}\sin t\,dt$. Let $u = e^{-st}$ and $v = -\cos t$. Then $I(s) = \displaystyle\int u\,dv = uv - \int v\,du =$ $-e^{-st}\cos t - sJ(s)$, where $J(s) = \displaystyle\int e^{-st}\cos t\,dt$. Now let $u_1 = e^{-st}$ and $v_1 = \sin t$. Then $J(s) = \displaystyle\int u_1\,dv_1 =$ $u_1 v_1 - \displaystyle\int v_1\,du_1 = e^{-st}\sin t + s\int e^{-st}\sin t\,dt = e^{-st}\sin t + sI$. So we have $I(s) = -e^{-st}\cos t - sJ(s) =$ $-e^{-st}\cos t - s\big(e^{-st}\sin t + sI(s)\big)$. We solve this for $I(s)$ to get $I(s) = \dfrac{-e^{-st}}{1+s^2}\big(\cos t + s\,\sin t\big)$. For fixed $s > 0$ we have $\displaystyle\lim_{t\to\infty} e^{-st}\big(\cos t + s\sin t\big) = 0$ and so $F(s) = \Big[I(s)\Big]_0^\infty = \dfrac{1}{1+s^2}$.

5: Evaluate the following improper integrals.

(a) $\displaystyle\int_2^\infty \frac{dx}{x^4\sqrt{x^2-4}}$

Solution: Write $I = \displaystyle\int_2^\infty \frac{dx}{x^4\sqrt{x^2-4}}$. Let $2\sec\theta = x$ so $2\tan\theta = \sqrt{x^2-4}$ and $2\sec\theta\tan\theta\,d\theta = dx$. Then $I = \displaystyle\int_0^{\pi/2} \frac{2\sec\theta\tan\theta\,d\theta}{16\sec^4\theta\,2\tan\theta} = \frac{1}{16}\int_0^{\pi/2}\cos^3\theta\ d\theta = \frac{1}{16}\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\,d\theta = \frac{1}{16}\int_0^1 1-u^2\ du$, where $u=\sin\theta$, and so $I = \frac{1}{16}\Big[u - \frac{1}{3}u^3\Big]_0^1 = \frac{1}{24}$.

(b) $\displaystyle\int_{-\infty}^\infty \frac{x(x+1)}{(x^2+1)^2}\ dx.$

Solution: Write $I = \displaystyle\int_{-\infty}^\infty \frac{x(x+1)}{(x^2+1)^2}$. To get $\displaystyle\frac{x(x+1)}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Bx+D}{(x^2+1)^2}$ we need to have $A(x^3+x) + B(x^2+1) + Cx + D = x^2 + x$. Equate coefficients to get $A=0$, $B=1$, $A+C=1$ and $B+D=0$. Solve these to get $A=0$, $B=1$, $C=1$ and $D=-1$. Thus $I = \displaystyle\int_{-\infty}^\infty \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} - \frac{1}{(x^2+1)^2}\ dx$. We have $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+1} = \Big[\tan^{-1}x\Big]_{-\infty}^\infty = \pi$, and $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)^2} = \Big[-\frac{1}{2(x^2+1)}\Big]_{-\infty}^\infty = 0$, and to get $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$ we let $\tan\theta = x$ so $\sec\theta = \sqrt{1+x^2}$ and $\sec^2\theta\,d\theta = dx$, and then $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = \int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi}{2}$. Thus $I = \pi + 0 - \frac{\pi}{2} = \frac{\pi}{2}$.