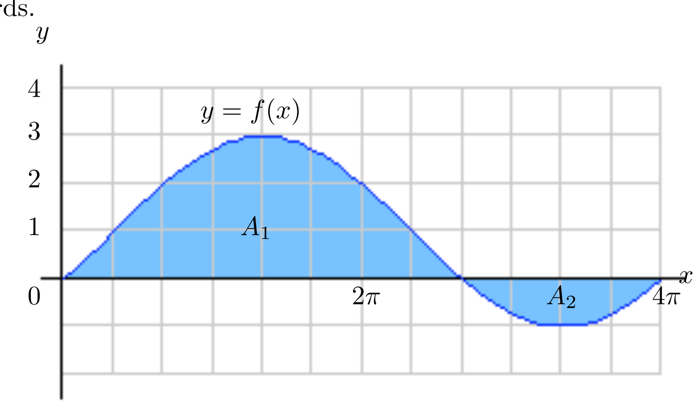
MATH 138 Calculus 2, Solutions to the Exercises for Chapter 4

1: Let $f(x) = 2\sin\left(\frac{1}{2}x - \frac{\pi}{6}\right) + 1$ for $0 \le x \le 4\pi$.

(a) Sketch the graph of $y = f(x)$ and shade the region which lies between the graph and the x-axis with $0 \le x \le 4\pi$ (one part of the region lies above the x-axis and one part lies below).

Solution: We have $f(x) = 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 1$, so we can obtain the graph of $y = f(x)$ from the graph of $y = \sin x$ by scaling by a factor of 2, both horizontally and vertically, then by translating $\frac{\pi}{3}$ units to the right and 1 unit upwards.



(b) Find the exact area of the region described in part (a).

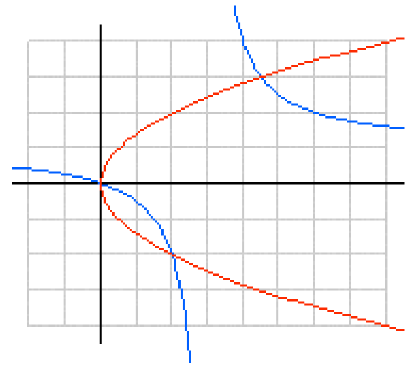
Solution: Let A_1 be the area of the part that lies above the x-axis and let A_2 be the area of the part which lies below. To find A_1 and A_2 we need to know the x-intercepts of the graph. A good graph (or table of values) will show the exact value of the x-intercepts, but they can also be found algebraically as follows. We have $f(x) = 0 \iff 2\sin\left(\frac{1}{2}x - \frac{\pi}{6}\right) + 1 = 0 \iff \sin\left(\frac{1}{2}x - \frac{\pi}{6}\right) = -\frac{1}{2} \iff \left(\frac{1}{2}x - \frac{\pi}{6}\right) = \frac{7\pi}{6}, \frac{11\pi}{6} \iff \frac{1}{2}x = \frac{4\pi}{3}, 2\pi \iff x = \frac{8\pi}{3}, 4\pi$. Now we can calculate the areas A_1 and A_2. By inspection (or by trial and error), an antiderivative for $f(x)$ is $-4\cos\left(\frac{1}{2}x - \frac{\pi}{6}\right) + x$, so we have

$$\begin{aligned}
A_1 &= \int_0^{8\pi/3} f(x)\,dx = \Big[-4\cos\left(\tfrac{1}{2}x - \tfrac{\pi}{6}\right) + x\Big]_0^{8\pi/3} = \left(-4\cos\left(\tfrac{7\pi}{6}\right) + \tfrac{8\pi}{3}\right) - \left(-4\cos\left(-\tfrac{\pi}{6}\right)\right) \\
&= \left(2\sqrt{3} + \tfrac{8\pi}{3}\right) + \left(2\sqrt{3}\right) = 4\sqrt{3} + \tfrac{8\pi}{3}\ , \text{ and} \\
A_2 &= \int_{8\pi/3}^{4\pi} -f(x)\,dx = \Big[4\cos\left(\tfrac{1}{2}x - \tfrac{\pi}{6}\right) - x\Big]_{8\pi/3}^{4\pi} = \left(4\cos\left(\tfrac{11\pi}{6}\right) - 4\pi\right) - \left(4\cos\left(\tfrac{7\pi}{6}\right) - \tfrac{8\pi}{3}\right) \\
&= \left(2\sqrt{3} - 4\pi\right) - \left(-2\sqrt{3} - \tfrac{8\pi}{3}\right) = 4\sqrt{3} - \tfrac{4\pi}{3}
\end{aligned}$$

The total area of the region is $A = A_1 + A_2 = 8\sqrt{3} + \frac{4\pi}{3}$.

2: Find the area of the region bounded by the curves $y^2 = 2\,x$ and $y = \dfrac{x}{x-3}$.

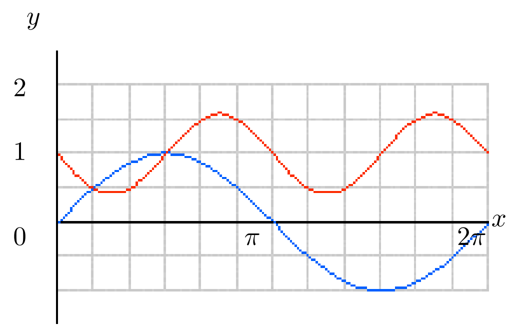
Solution: First sketch the two curves. The hyperbola $y = \dfrac{x}{x-3}$ is shown at right in blue, and the parabola $y^2 = 2x$ is in red. We see that the region in question lies below the hyperbola and above the bottom half of the parabola, which is given by $y = -\sqrt{2x}$, with $0 \le x \le 2$. Thus the area is



$$\begin{aligned} A &= \int_0^2 \left(\frac{x}{x-3}\right) - \left(-\sqrt{2x}\right)\,dx \\ &= \int_0^2 \frac{x-3+3}{x-3} + \sqrt{2x}\,dx \\ &= \int_0^2 1 + \frac{3}{x-3} + (2x)^{1/2}\,dx \\ &= \Big[x + 3\ln\big|x-3\big| + \tfrac{1}{3}(2x)^{3/2}\Big]_0^2 \\ &= \big(2 + 0 + \tfrac{8}{3}\big) - \big(3\ln 3\big) \\ &= \tfrac{14}{3} - 3\ln 3\,. \end{aligned}$$

3: Find the area of the region bounded by the curves $y = \sin x$ and $y = 1 - \frac{1}{\sqrt{3}}\sin(2x)$ between their two points of intersection with $0 \le x \le 2\pi$.

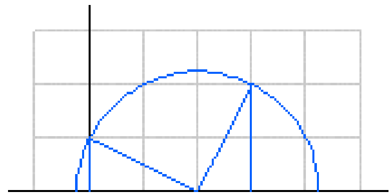
Solution: First we sketch the two curves. The curve $y = \sin x$ is shown in blue and the curve $y = 1 - \frac{1}{\sqrt{3}}\sin 2x$ is shown in red.



The region lies below the curve $y = \sin x$ and above the curve $y = 1 - \frac{1}{\sqrt{3}}\sin 2x$ with $\frac{\pi}{6} \le x \le \frac{\pi}{2}$, so the area is given by

$$\begin{aligned} A &= \int_{\pi/6}^{\pi/2} \sin x - \big(1 - \tfrac{1}{\sqrt{3}}\sin 2x\big)\,dx \\ &= \int_{\pi/6}^{\pi/2} \sin x + \tfrac{\sqrt{3}}{3}\sin 2x - 1\,dx \\ &= \Big[-\cos x - \tfrac{\sqrt{3}}{6}\cos 2x - x\Big]_{\pi/6}^{\pi/2} \\ &= \big(0 - \tfrac{\sqrt{3}}{6}(-1) - \tfrac{\pi}{2}\big) - \big(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{6}\cdot\tfrac{1}{2} - \tfrac{\pi}{6}\big) \\ &= \sqrt{3}\big(\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{12}\big) - \pi\big(\tfrac{1}{2} - \tfrac{1}{6}\big) \\ &= \tfrac{3\sqrt{3}}{4} - \tfrac{\pi}{3}\,. \end{aligned}$$

4: A rod of length 3 m lies along the axis with one end at $x=0$ and the other end at $x=3$. The linear densityat each point, in kg/m, is given by $\rho(x)=\sqrt{1+4x-x^2}$. Find the total mass and the average linear density of the rod.



Solution: The total mass is given by $m=\int_0^3 \rho(x)\,dx=\int_0^3 \sqrt{1+4x-x^2}\,dx=\int_0^3 \sqrt{5-(x-2)^2}\,dx$. This integral can be evaluated using the substitution $\sqrt{5}\sin\theta=x-2$, but it is much easier to notice that the region under the circle $y=\sqrt{5-(x-2)^2}$ between $x=0$ and $x=3$ can be cut up into a quarter-circle, which has area $\frac{1}{4}\pi\left(\sqrt{5}\right)^2$, and two triangles, each of area 1. Thus the mass is $m=\int_0^3 \sqrt{5-(x-2)^2}\,dx=2+\frac{5\pi}{4}$. The average density is $\overline{\rho}=\frac{1}{3}\,m=\frac{2}{3}+\frac{5\pi}{12}$.

5: (a) Let R be the region given by $0\le y\le 1-\frac{1}{4}\,x^2$ and $-2\le x\le 2$. Find the volume of the solid obtained by revolving R about the x-axis.

Solution: The volume is $V=2\int_0^2 \pi\left(1-\frac{1}{4}\,x^2\right)^2\,dx=2\pi\int_0^2 1-\frac{1}{2}\,x^2+\frac{1}{16}\,x^4\,dx=2\pi\left[x-\frac{1}{6}\,x^3+\frac{1}{80}\,x^5\right]_0^2=2\pi\left(2-\frac{4}{3}+\frac{2}{5}\right)=\frac{32}{15}\,\pi$.

(b) Let S be the region given by $\frac{1}{4}\,x^2-1\le y\le 1-\frac{1}{4}\,x^2$ and $0\le x\le 2$. Find the volume of the solid obtained by revolving S about the y-axis.

Solution: Using cylindrical shells, the volume is given by $V=2\int_0^2 2\pi x\left(1-\frac{1}{4}\,x^2\right)\,dx=4\pi\int_0^2 x-\frac{1}{4}\,x^2\,dx=4\pi\left[\frac{1}{2}\,x^2-\frac{1}{16}\,x^4\right]_0^2=4\pi(2-1)=4\pi$.

6: Let R be the (infinitely long) region given by $0\le y\le \dfrac{1}{1+x^2}$ and $x\ge 0$.

(a) Find the volume of the solid obtained by revolving R about the x-axis.

Solution: The volume is given by $V=\int_{x=0}^{\infty}\pi\,\dfrac{1}{(1+x^2)^2}\,dx$. We let $\tan\theta=x$ so $\sec\theta=\sqrt{1+x^2}$ and $\sec^2\theta\,d\theta=dx$, and then we obtain $V=\int_{\theta=0}^{\pi/2}\pi\,\dfrac{\sec^2\theta\,d\theta}{\sec^4\theta}=\pi\int_0^{\pi/2}\cos^2\theta\,d\theta=\pi\int_0^{\pi/2}\frac{1}{2}+\frac{1}{2}\cos 2\theta\;d\theta=\pi\left[\frac{1}{2}\,\theta+\frac{1}{4}\,\sin 2\theta\right]_0^{\pi/2}=\frac{\pi^2}{4}$.

(b) Find the volume of the solid obtained by revolving R about the y-axis.

Solution: Using cylindrical shells, the volume is $V=\int_{x=0}^{\infty}2\pi\,x\,\dfrac{1}{1+x^2}\,dx$. Let $u=1+x^2$ so that $du=2x\,dx$, and then $V=\pi\int_{u=1}^{\infty}\dfrac{1}{u}\,du=\pi\Big[\ln u\Big]_1^{\infty}=\infty$

7: Find the volume of the solid which is obtained by revolving the disc $(x-1)^2+y^2\le 1$ about the y-axis.

Solution: Using cylindrical shells, the volume is $V=2\int_{x=0}^{2}2\pi\,x\sqrt{1-(x-1)^2}\,dx$. Let $\sin\theta=x-1$ so that $\cos\theta=\sqrt{1-(x-1)^2}$ and $\cos\theta\,d\theta=dx$. Then we have $V=4\pi\int_{\theta=-\pi/2}^{\pi/2}(\sin\theta+1)\cos\theta\,\cos\theta\,d\theta=4\pi\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta+\cos^2\theta\,d\theta=4\pi\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta=4\pi\Big[-\frac{1}{3}\cos^3\theta+\frac{1}{2}\,\theta+\frac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/2}=4\pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=2\pi^2$.

8: A circular hole of radius 1 is bored through the center of a wooden ball of radius 2. Find the volume of the remaining portion of the ball.

Solution: We provide two solutions. For the first solution, we note that the remaining portion of the ball is in the shape of the solid obtained by revolving the region given by $1\le y\le\sqrt{4-x^2}$ and $-\sqrt{3}\le x\le\sqrt{3}$ about the x-axis. The cross-section at x is shaped like an annulus (that is a circular disc with a smaller circular hole in the center) with outer radius $\sqrt{4-x^2}$ and inner radius 1. The cross-sectional area is $A(x)=\pi(4-x^2)-\pi=\pi(3-x^2)$. The volume is $V=2\int_0^{\sqrt{3}}\pi(3-x^2)\,dx=2\pi\Big[3x-\frac{1}{3}\,x^3\Big]_0^{\sqrt{3}}=2\pi\left(3\sqrt{3}-\sqrt{3}\right)=4\pi\sqrt{3}$.

For the second solution, we note that the remaining portion of the ball is in the shape of the solid obtained by revolving the region given by $-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$ and $1\le x\le 2$ about the y-axis, so using cylindrical shells, the volume is $V=2\int_{x=1}^{2}2\pi\,x\sqrt{4-x^2}\,dx$. Letting $u=4-x^2$ so that $du=-2x\,dx$ we obtain $V=\int_{u=3}^{0}-2\pi\sqrt{u}\,du=-2\pi\Big[\frac{2}{3}\,u^{3/2}\Big]_3^0=-2\pi\left(-2\sqrt{3}\right)=4\pi\sqrt{3}$.

9: Find the arclength of the curve $y=e^x$ with $0\le x\le\ln 2$.

Solution: We have $y'=e^x$ so that arclength is $L=\int_{x=0}^{\ln 2}\sqrt{1+e^{2x}}\,dx$. Let $u=\sqrt{1+e^{2x}}$ so that $u^2=1+e^{2x}$ and $2u\,du=2e^{2x}\,dx$, so $dx=\frac{u}{e^{2x}}\,du=\frac{u}{u^2-1}\,du$. Then we obtain $L=\int_{u=\sqrt{2}}^{\sqrt{5}}\frac{u^2\,du}{u^2-1}=\int_{\sqrt{2}}^{\sqrt{5}}1+\frac{1}{u^2-1}\,du=\int_{\sqrt{2}}^{\sqrt{5}}1+\frac{\frac{1}{2}}{u-1}-\frac{\frac{1}{2}}{u+1}\,du=\Big[u+\frac{1}{2}\,\ln\Big|\frac{u-1}{u+1}\Big|\Big]_{\sqrt{2}}^{\sqrt{5}}=\sqrt{5}+\frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)-\sqrt{2}-\frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$.

10: Find the arclength of the portion of the parabola $y=x^2$ with $0\le x\le 1$.

Solution: We have $y'=2x$ so $\sqrt{1+(y')^2}=\sqrt{1+4x^2}$. Let $\tan\theta=2x$, $\sec\theta=\sqrt{1+4x^2}$, $\sec^2\theta\,d\theta=2\,dx$. Then the arclength is $L=\int_0^1\sqrt{1+4x^2}\,dx=\int_{x=0}^{1}\frac{1}{2}\sec^3\theta\,d\theta=\Big[\frac{1}{4}\sec\theta\tan\theta+\frac{1}{4}\ln|\sec\theta+\tan\theta|\Big]_{x=0}^{1}=\Big[\frac{1}{2}x\sqrt{1+4x^2}+\frac{1}{4}\ln\big|2x+\sqrt{1+4x^2}\big|\Big]_0^1=\frac{\sqrt{5}}{2}+\frac{1}{4}\ln(2+\sqrt{5})$.

11: Find the area of the surface which is obtained by revolving the portion of the cubic curve $y=x^3$ with $0\le x\le 1$ about the y-axis.

Solution: We have $y'=3x^2$ so $\sqrt{1+(y')^2}=\sqrt{1+9x^4}$. Let $\tan\theta=3x^2$ so that $\sec\theta=\sqrt{1+9x^4}$ and $\sec^2\theta\,d\theta=6x\,dx$. Then the surface area is $A=\int_0^1 2\pi x\sqrt{1+9x^4}\,dx=\frac{\pi}{3}\int_{x=0}^{1}\sec^3\theta\,d\theta=\frac{\pi}{3}\Big[\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\Big]_{x=0}^{1}=\frac{\pi}{6}\Big[3x^2\sqrt{1+9x^4}+\ln\big|3x^2+\sqrt{1+9x^4}\big|\Big]_0^1=\frac{\pi}{6}\big(3\sqrt{10}+\ln(3+\sqrt{10})\big)$.