MATH 138 Calculus 2, Solutions to the Exercises for Chapter 5

1: Find a solution of the form $y = a + bx + cx^2$ to the differential equation $y'' + (1 + x)y' - 2y = 5x - 3$ with $y(1) = 4$.

Solution: Let $y = a + bx + cx^2$. Then $y(1) = a + b + c$ so to get $y(1) = 4$ we need $a + b + c = 4 \quad (1)$. Also, we have $y' = b + 2c\,x$ and $y'' = 2c$ so $y'' + (1 + x)y' - 2y = 2c + (1 + x)(b + 2c\,x) - 2(a + bx + cx^2) = (2c+b-2a)+(2c+b-2b)x+(2c-2c)x^2 = (-2a+b+2c)+(-b+2c)x$ and so to get $y''+(1+x)y'-2y = 5x-3$ we need $-b + 2c = 5 \quad (2)$ and $-2a + b + 2c = -3 \quad (3)$. We solve equations (1), (2) and (3):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & 2 & 5 \\ -2 & 1 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & -5 \\ 0 & 3 & 5 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 10 & 20 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 9 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We find that $(a, b, c) = (3, -1, 2)$ and so the solution to the differential equation is $y = 3 - x + 2x^2$.

2: Solve the following differential equations.

(a) $x\,y' + 2y = \sqrt{1 + x^2}$

Solution: This DE is linear since we can write it in the form $y' + \frac{2}{x}y = \frac{\sqrt{1+x^2}}{x}$. An integrating factor is $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and the solution is $y = \frac{1}{x^2}\int x\sqrt{1+x^2}\,dx = \frac{1}{x^2}\big(\frac{1}{3}(1+x^2)^{3/2} + c\big)$.

(b) $y' = e^{x+2y}$

Solution: This DE is separable since we can write it as $e^{-2y}\,dy = e^x\,dx$. Integrate both sides to get $\int e^{-2y}\,dy = \int e^x\,dx \Longrightarrow -\frac{1}{2}e^{-2y} = e^x - \frac{1}{2}c \Longrightarrow e^{-2y} = c - 2e^x \Longrightarrow -2y = \ln\left|c - 2e^x\right| \Longrightarrow y = -\frac{1}{2}\ln\left|c - 2e^x\right|$.

3: Solve the following initial value problems.

(a) $(1 + x^2)y' = x\,y$ with $y(0) = 2$.

Solution: This DE is separable. We write it as $\frac{dy}{y} = \frac{x}{1+x^2}$ then integrate to get $\ln|y| = \frac{1}{2}\ln(1 + x^2) + c$ or $y = A\,e^{\frac{1}{2}\ln(1+x^2)} = A\sqrt{1+x^2}$. The condition $y(0) = 2$ then gives $2 = A$ and so $y = 2\sqrt{1+x^2}$.

(b) $x^2y' - y = 1$ with $y(1) = 1$.

Solution: This DE is linear; we can write it as $y' - \frac{1}{x^2}\,y = \frac{1}{x^2}$. An integrating factor is $\lambda = e^{\int -\frac{1}{x^2}\,dx} = e^{1/x}$ and the solution to the DE is $y = e^{-1/x}\int \frac{1}{x^2}\,e^{1/x}\,dx = e^{-1/x}\big(-e^{1/x} + c\big) = -1 + c\,e^{-1/x}$. Put in $y(1) = 1$ to get $1 = -1 + c/e$ so $c = 2e$ and the solution to the IVP is $y = -1 + 2\,e^{1-1/x}$.

4: (a) Solve the initial value problem $y'' - 2y' = x$ with $y(0) = 0$ and $y'(0) = 0$.
(Hint: first let $u(x) = y'(x)$ so that $y''(x) = u'(x)$ and solve the resulting DE for $u = u(x)$).

Solution: The dependent variable y does not appear in this DE so we let $u = y'$ and we have $y'' = u'$, so the DE becomes $u' - 2u = x$. This is linear; an integrating factor is $\lambda = e^{\int -2\,dx} = e^{-2x}$ and, using Integration by Parts, the solution is $u = e^{2x}\int x\,e^{-2x}\,dx = e^{2x}\big(-\frac{1}{2}\,x\,e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx\big) = e^{2x}\big(-\frac{1}{2}\,x\,e^{-2x} - \frac{1}{4}\,e^{-2x} + a\big) = -\frac{1}{2}\,x - \frac{1}{4} + a\,e^{2x}$. Put in $y(0) = 0$ and $u(0) = y;(0) = 0$ to get $0 = a - \frac{1}{4}$ so $a = \frac{1}{4}$ and we have $y' = u = -\frac{1}{2}\,x - \frac{1}{4} + \frac{1}{4}\,e^{2x}$. Integrate to get $y = -\frac{1}{4}\,x^2 - \frac{1}{4}\,x + \frac{1}{8}\,e^{2x} + c$. Put in $y(0) = 0$ to get $0 = \frac{1}{8} + c$, so $c = -\frac{1}{8}$ and the solution to the IVP is $y = -\frac{1}{4}\,x^2 - \frac{1}{4}\,x + \frac{1}{8}\,e^{2x} - \frac{1}{8}$.

(b) Solve the initial value problem $y'' = (1 - 2y)\,y'$ with $y(0) = 1$ and $y'(0) = 2$.
(Hint: first let $u(y(x)) = y'(x)$ so that $u'(y(x))y'(x) = y''(x)$ and solve the resulting DE for $u = u(y)$).

Solution: The independent variable x does not appear in the DE so we let $u = y'$ and we have $y'' = u\,u'$, so the DE becomes $u\,u' = (1 - 2y)\,u$ or $u' = 1 - 2y$. Integrate to get $u = y - y^2 + a$. Put in $y(0) = 1$ and $u(0) = y'(0) = 2$ to get $2 = 1 - 1 + a$ so $a = 2$, and so we have $u = y - y^2 + 2$, that is $y' = -y^2 + y + 2 = -(y - 2)(y + 1)$. This is separable; we write it as $\frac{dy}{(y-2)(y+1)} = -dx$ and integrate (using partial fractions) to get $\frac{1}{3}\ln\left|\frac{y-2}{y+1}\right| = -x + \frac{1}{3}b$, that is $\ln\left|\frac{y-2}{y+1}\right| = b - 3x$ or $\frac{y-2}{y+1} = c\,e^{-3x}$. Put in $y(0) = 1$ to get $-\frac{1}{2} = c$, so we have $\frac{y-2}{y+1} = -\frac{1}{2}e^{-3x}$. Solve this for y to get $y = \frac{2-\frac{1}{2}e^{-3x}}{1+\frac{1}{2}e^{-3x}} = \frac{4\,e^{3x}-1}{2\,e^{3x}+1}$.

5: The amount $A(t)$ of a radioactive substance decays exponentially with a half-life of 3 seconds. If $A(2) = 20$ then find the time t at which $A(t) = 4$.

Solution: $A(t)$ satisfies the DE $A' = kA$ for some $k < 0$. Solve this DE to get $A(t) = A(0)e^{kt}$. Since the half life is 3 seconds, we have $A(3) = \frac{1}{2}A(0)$, so $A(0)e^{3k} = \frac{1}{2}A(0)$ hence $e^{3k} = \frac{1}{2}$. Solve this for k to get $k = \frac{1}{3}\ln\frac{1}{2} = -\frac{1}{3}\ln 2$. Since $A(2) = 20$ we have $A(0)e^{2k} = 20$, so $A(0) = 20e^{-2k}$, and so in general we have $A(t) = 20e^{-2k}e^{kt} = 20e^{k(t-2)}$. We have $e(t) = 4 \iff 20e^{k(t-2)} = 4 \iff e^{k(t-2)} = \frac{1}{5} \iff k(t-2) = \ln(1/5) = -\ln 5 \iff t = 2 - \dfrac{\ln 5}{k} = 2 + \dfrac{3\ln 5}{\ln 2}$.

6: A pot of boiling water is removed from the heat and placed on a table in a room. After 2 minutes, the water has cooled from 100° to 84°. After another 2 minutes, it has cooled to 72°. What is the temperature in the room?

Solution: Let $T(t)$ be the temperature of the water at time t, and let K be the temperature of the room. Then (according to Newton's Law of Cooling) $T' = k(T-K)$ for some $k < 0$. This DE is separable, since we can write it as $\dfrac{dT}{T-K} = k\,dt$. Integrate both sides to get $\ln|T-K| = kt + c$, so when $T > K$ we have $T = K + Ae^{kt}$ where $A = e^c$. The conditions $T(0) = 100$, $T(2) = 84$ and $T(4) = 72$ give us the three equations $K + A = 100$ (1), $K + Ae^{2k} = 84$ (2), and $K + Ae^{4t} = 72$ (3). Equation (3) gives $Ae^{4k} = 72 - K$ (4) and equation (2) gives $A^2e^{4k} = (84-K)^2$ (5). Divide (5) by (4) to get $A = \dfrac{(84-K)^2}{72-K}$. By (1) we have $A = 100 - K$ so $(100-K)(72-K) = (84-K)^2$, that is $7200 - 172K + K^2 = 7056 - 168K + K^2$, so $4K = 144$. Thus the room temperature is $K = 36$.

7: Water drains from a hole of area 25 cm^2 at the bottom tip of a conical tank of radius 1 m and height 4 m. If the water drains at a velocity of $v = 4\sqrt{y}\ \ m/s$, where $y\ m$ is the depth of the water in the tank, then find the time at which the tank will be empty.

Solution: When the depth of the water is y, the surface of the water is a circle of radius $\frac{1}{4}y$, which has area $A(y) = \frac{\pi}{16}y^2$. The change in volume of the water in the tank in the time interval dt is given by $dV = A(y)\,dy = \frac{\pi}{16}y^2\,dy$, and also by $dV = -av\,dt = -\frac{1}{400}4\sqrt{y}\ dt = -\frac{1}{100}\sqrt{y}\ dt$. So y satisfies the DE $\frac{\pi}{16}y^2 = -\frac{1}{100}\sqrt{y}$ which we can write as $y^{\frac{3}{2}}\,dy = -\frac{4}{25\pi}\,dt$. Integrate both sides to get $\frac{2}{5}y^{5/2} = -\frac{4}{25\pi}t + c$. Put in $y(0) = 4$ to get $c = \frac{2}{5}\cdot 32$, so we have $\frac{2}{5}y^{5/2} = \frac{2}{5}\cdot 32 - \frac{4}{25\pi}t$, that is $y = \left(32 - \frac{2}{5\pi}t\right)^{2/5}$. Thus we have $y = 0$ when $t = 80\pi$.

8: A tank contains 100 L of water. A solution with salt concentration 0.5 kg/L is added at 6 L/min. The solution is kept well mixed and is drained from the tank at a rate of 4 L/min. Find the concentration of salt in the tank when it contains 200 L of solution.

Solution: Let $V(t)$ be the volume of liquid in the tank at time t, let $S(t)$ be the amount of salt in the tank, and let r_{in}, r_{out}, c_{in} and c_{out} be the incoming and outgoing rates and concentrations. Then we have $V(t) = 100 + (r_{in} - r_{out})t = 100 + 2t$, and $S'(t) = r_{in}c_{in} - r_{out}c_{out} = 6\,\frac{1}{2} - 4\,\frac{S}{V} = 3 - \frac{2S}{50+t}$. We write this as $S' + \frac{2}{50+t}S = 3$, which is linear. An integrating factor is $e^{\int 2/(50+t)\,dt} = e^{2\ln(50+t)} = (50+t)^2$, and the solution to the DE is $S = (50+t)^{-2}\int 3(50+t)\,dt = (50+t)^{-2}\big((50+t)^3 + c\big) = 50 + t + c(50+t)^{-2}$. The initial condition $S(0) = 0$ gives $50 + c/(50)^2 = 0$ so $c = -(50)^3$, and we have $S = 50 + t - (50)^3(50+t)^{-2}$. The tank contains 200 L when $t = 50$, and we have $S(50) = 100 - (50)^3(100)^{-2} = 87.5$, and the concentration is $C = S(50)/200 = \frac{875}{2000} = \frac{7}{16}$

9: Let $x(t)$ be the height of an object of mass m which is thrown upwards from the ground. If the force of air resistance is $-kx'$, then $x(t)$ satisfies the DE $mx'' + kx' + mg = 0$. Suppose that $m = 1$, $k = \frac{1}{10}$, $g = 10$, $x(0) = 0$ and $x'(0) = 20$. Find the time t at which the object reaches its maximum height.

Solution: The dependent variable x does not occur in the DE so we write $x' = v$ and we have $x'' = v'$. The DE becomes $v' + \frac{1}{10}v = -10$. This is linear. An integrating factor is $\lambda = e^{\int \frac{1}{10}\,dt} = e^{t/10}$, and the solution to the DE is $v = e^{-t/10}\int -10e^{t/10}\,dt = e^{-t/10}\big(-100\,e^{t/10} + c\big) = c\,e^{-t/10} - 100$. The initial condition $v(0) = 20$ gives $c - 100 = 20$ so $c = 120$ and we have $v = 120\,e^{-t/10} - 100$. The object will reach its maximum height when $v = 0$, so we solve $120\,e^{-t/10} = 100$ to get $t = 10\ln\frac{6}{5}$.