MATH 138 Calculus 2, Solutions to the Exercises for Chapter 7

1: Find the limit of each of the following sequences $\{a_n\}$, if the limit exists.

(a) $a_n = \dfrac{\sqrt{4n^2+3}}{n-\sqrt{n}}$

Solution: $a_n = \dfrac{\sqrt{4n^2+3}}{n-\sqrt{n}} = \dfrac{\sqrt{4+\frac{3}{n^2}}}{1-\frac{1}{\sqrt{n}}} \longrightarrow \dfrac{\sqrt{4}}{1} = 2.$

(b) $a_n = \dfrac{(-3)^n}{2^{2n+1}}$

Solution: $a_n = \dfrac{(-3)^n}{2^{2n+1}} = \frac{1}{2}\left(-\frac{3}{4}\right)^n \longrightarrow 0$ since $\left|-\frac{3}{4}\right| < 1$.

(c) $a_n = \dfrac{2^{2n}}{n!}$

Solution: Note that $a_n = \dfrac{2^{2n}}{n!} = \dfrac{4^n}{n!} = \dfrac{4\cdot 4\cdot 4\cdots 4}{1\cdot 2\cdot 3\cdots n} = \frac{4}{1}\cdot\frac{4}{2}\cdot\frac{4}{3}\cdot\left(\frac{4}{4}\right)\left(\frac{4}{5}\right)\left(\frac{4}{6}\right)\cdots\left(\frac{4}{n-1}\right)\cdot\frac{4}{n} \le \frac{4}{1}\cdot\frac{4}{2}\cdot\frac{4}{3}\cdot\frac{4}{n} = \frac{128}{3n}$, since all the terms in brackets are ≤ 1. Since $0 \le a_n \le \frac{128}{3n}$ and $\frac{128}{3n} \longrightarrow 0$, we have $a_n \longrightarrow 0$ by the Squeeze Theorem.

(d) $a_n = \left(\dfrac{n+1}{n-1}\right)^n$.

Solution: $a_n = e^{n\ln\left(\frac{n+1}{n-1}\right)} \longrightarrow e^2$ since $\displaystyle\lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n-1}{n+1}\,\frac{(n-1)-(n+1)}{(n-1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{2n^2}{n^2-1} \longrightarrow 2$, where we used l'Hôpital's Rule (treating n as a real variable).

2: (a) Let $a_1 = \frac{4}{3}$ and $a_{n+1} = 5 - \dfrac{4}{a_n}$ for $n \geq 1$. Determine whether $\{a_n\}$ converges and, if so, find the limit.

Solution: If $\{a_n\}$ does converge, say $a_n \to l$, then we also have $a_{n+1} \to l$, and so taking the limits on both sides of the formula $a_{n+1} = 5 - \frac{4}{a_n}$ gives $l = 5 - \frac{4}{l} \implies l^2 = 5l - 4 \implies l^2 - 5l + 4 = 0 \implies (l-1)(l-4) = 0$. This shows that if the limit exists then it must be equal to 1 or 4.

The first few terms of the sequence are $a_1 = \frac{4}{3}$, $a_2 = 2$ and $a_3 = 3$. Since the terms appear to be increasing, we shall try to prove that $1 \leq a_n \leq a_{n+1} \leq 4$ for all $n \geq 1$. This is true when $n = 1$. Suppose it is true when $n = k$. Then we have $1 \leq a_k \leq a_{k+1} \leq 4 \implies 1 \geq \frac{1}{a_k} \geq \frac{1}{a_{k+1}} \geq \frac{1}{4} \implies -4 \leq -\frac{4}{a_k} \leq -\frac{4}{a_{k+1}} \leq -1$ $\implies 1 \leq 5 - \frac{4}{a_k} \leq 5 - \frac{4}{a_{k+1}} \leq 4$, that is $1 \leq a_{k+1} \leq a_{k+2} \leq 4$. Thus, by mathematical induction, we have $1 \leq a_n \leq a_{n+1} \leq 4$ for all $n \geq 1$.

Since $a_n \leq a_{n+1}$ for all $n \geq 1$, the sequence is increasing, and since $a_n \leq 4$ for all $n \geq 1$, the sequence is bounded above. Thus the sequence does converge. Since we know the limit must be either 1 or 4, and since the sequence starts at $a_1 = 2$ and increases, the limit must be 4.

(b) Let $a_1 = 2$ and $a_{n+1} = \sqrt{3{a_n}^2 - 3}$ for $n \geq 1$. Determine whether $\{a_n\}$ converges and, if so, find the limit.

Solution: If $\{a_n\}$ does converge, say $a_n \to l$, then we also have $a_{n+1} \to l$, and so taking the limit on both sides of the formula $a_{n+1} = \sqrt{3{a_n}^2 - 3}$ gives $l = \sqrt{3l^2 - 3} \implies l^2 = 3l^2 - 3 \implies 2l^2 = 3 \implies l = \pm\sqrt{\frac{3}{2}}$. Only the positive value is a solution to $l = \sqrt{3l^2 - 3}$, so if the limit exists then it must be $\sqrt{\frac{3}{2}}$.

The first few terms are $a_1 = 2$, $a_2 = \sqrt{9} = 3$ and $a_3 = \sqrt{24} = 2\sqrt{6}$. Since the sequence appears to be increasing, we shall try to prove that $\sqrt{\frac{3}{2}} \leq a_n \leq a_{n+1}$ for all n. This is true when $n = 1$. Suppose it is true when $n = k$. Then we have $\sqrt{\frac{3}{2}} \leq a_k \leq a_{k+1} \implies \frac{3}{2} \leq {a_k}^2 \leq {a_{k+1}}^2 \implies \frac{9}{2} \leq 3\,{a_k}^2 \leq 3\,{a_{k+1}}^2 \implies$ $\frac{3}{2} \leq 3\,{a_k}^2 - 3 \leq 3\,{a_{k+1}}^2 - 3 \implies \sqrt{\frac{3}{2}} \leq \sqrt{3\,{a_k}^2 - 3} \leq \sqrt{3\,{a_{k+1}}^2 - 3}$, that is $\sqrt{\frac{3}{2}} \leq a_{k+1} \leq a_{k+2}$. Thus, by mathematical induction, we have $\sqrt{\frac{3}{2}} \leq a_n \leq a_{n+1}$ for all $n \geq 1$.

Since the sequence starts at $a_1 = 2$ and increases, the limit cannot possibly be $\sqrt{\frac{3}{2}}$, so the sequence diverges to infinity.

3: (a) Find $\displaystyle\sum_{n=1}^{\infty}\frac{1+2^n}{2^{2n+1}}$, if it exists.

Solution: $\displaystyle\sum_{n=1}^{\infty}\frac{1+2^n}{2^{2n+1}}=\sum_{n=1}^{\infty}\frac{1}{2^{2n+1}}+\sum_{n=1}^{\infty}\frac{2^n}{2^{2n+1}}=\frac{\frac{1}{8}}{1-\frac{1}{4}}+\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$.

(b) Find $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+4n+3}$, if it exists.

Solution: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+4n+3}=\sum_{n=0}^{\infty}\frac{\frac{1}{2}}{n+1}-\frac{\frac{1}{2}}{n+3}$. The l^{th} partial sum is $S_l=\frac{1}{2}\displaystyle\sum_{n=0}^{l}\frac{1}{n+1}-\frac{1}{n+3}=$ $\frac{1}{2}\Big(\big(\frac{1}{1}-\frac{1}{3}\big)+\big(\frac{1}{2}-\frac{1}{4}\big)+\big(\frac{1}{3}-\frac{1}{5}\big)+\big(\frac{1}{4}-\frac{1}{6}\big)+\cdots+\big(\frac{1}{l-2}+\frac{1}{l}\big)+\big(\frac{1}{l-1}-\frac{1}{l+1}\big)+\big(\frac{1}{l}-\frac{1}{l+2}\big)+\big(\frac{1}{l+1}-\frac{1}{l+3}\big)\Big)=$ $\frac{1}{2}\big(1+\frac{1}{2}-\frac{1}{l+2}-\frac{1}{l+3}\big)\to\frac{1}{2}\big(1+\frac{1}{2}\big)=\frac{3}{4}$ as $l\to\infty$. Thus the sum is $\frac{3}{4}$.

(c) A hypothetical ball bounces as follows: when it is in the air, it has a constant downwards acceleration of $g=10$; when it bounces, it rebounds instantaneously; whenever it drops from a height h, it rebounds to a height of $\frac{3}{4}h$. This ball is dropped from an initial height $h=5$ and allowed to bounce indefinitely. Find the total distance travelled by the ball, and determine how long it takes for the ball to come to rest.

Solution: More generally, if the ball is dropped form an initial height h, then it falls a distance h, rebounds and climbs a distance $\frac{3}{4}h$ and falls the same distance $\frac{3}{4}h$, then rebounds and climbs $\big(\frac{3}{4}\big)^2h$ and falls the same distance, then rebounds and climbs $\big(\frac{3}{4}\big)^3h$ and falls the same distance, and so on. The total distance travelled is

$$d=h+2\big(\tfrac{3}{4}\big)h+2\big(\tfrac{3}{4}\big)^2h+2\big(\tfrac{3}{4}\big)^3h+\cdots=h\Big(1+2\big(\tfrac{3}{4}\big)+\big(\tfrac{3}{4}\big)^2+2\big(\tfrac{3}{4}\big)^3+\cdots\Big)=h\left(1+\tfrac{2\cdot\frac{3}{4}}{1-\frac{3}{4}}\right)=7\,h\,.$$

When the initial height is $h=5$, the total distance is $d=35$.

Since the acceleration is $a=-g$, when the ball is dropped at $t=0$ from an initial height $x(0)=h$ with an initial speed $v(0)=x'(0)=0$, the velocity is $v(t)=\int g\,dt=-gt+v(0)=-gt$, and the position is $x(t)=\int -gt\,dt=-\frac{1}{2}gt^2+x(0)=h-\frac{1}{2}gt^2$. The ball lands when $x(t)=0$, that is when $\frac{1}{2}gt^2=h$, or $t=\sqrt{\frac{2h}{g}}$. Thus the time taken for the ball to drop to the ground from a height of h is equal to $\sqrt{\frac{2h}{g}}$. Similarly, it takes the same amount of time from the moment the ball rebounds off the ground until the moment it reached a maximum height of h. Thus the total amount of time until the ball comes to rest is

$$\begin{aligned}
t&=\sqrt{\tfrac{2h}{g}}+2\sqrt{\tfrac{2\cdot\frac{3}{4}h}{g}}+2\sqrt{\tfrac{2\left(\frac{3}{4}\right)^2h}{g}}+2\sqrt{\tfrac{2\left(\frac{3}{4}\right)^3h}{g}}+\cdots\\
&=\sqrt{\tfrac{2h}{g}}\left(1+2\left(\sqrt{\tfrac{3}{4}}\right)+2\left(\sqrt{\tfrac{3}{4}}\right)^2+2\left(\sqrt{\tfrac{3}{4}}\right)^3+\cdots\right)\\
&=\sqrt{\tfrac{2h}{g}}\left(1+\tfrac{2\sqrt{\frac{3}{4}}}{1-\sqrt{\frac{3}{4}}}\right)\;=\sqrt{\tfrac{2h}{g}}\left(1+\tfrac{2\sqrt{3}}{2-\sqrt{3}}\right)\\
&=\sqrt{\tfrac{2h}{g}}\left(1+\tfrac{2\sqrt{3}}{2-\sqrt{3}}\cdot\tfrac{2+\sqrt{3}}{2+\sqrt{3}}\right)\;=\sqrt{\tfrac{2h}{g}}\left(1+4\sqrt{3}+6\right)\\
&=\sqrt{\tfrac{2h}{g}}\left(7+4\sqrt{3}\right)
\end{aligned}$$

When $h=5$ and $g=10$, the total time taken is $t=7+4\sqrt{3}$.

4: Determine which of the following series converge.

(a) $\sum \dfrac{n^2+4n}{\sqrt{n^5-2n+1}}$

Solution: Let $a_n = \dfrac{n^2+4n}{\sqrt{n^5-2n+1}}$ and let $b_n = \dfrac{n^2}{\sqrt{n^5}} = \dfrac{1}{\sqrt{n}}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$, and $\sum b_n$ diverges, and so $\sum a_n$ diverges too, by the Limit Comparison Test.

(b) $\sum \dfrac{n^4}{2^n}$

Solution: Let $a_n = \dfrac{n^4}{2^n}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = \lim\limits_{n\to\infty} \dfrac{(n+1)^4}{2^{n+1}} \dfrac{2^n}{n^4} = \frac{1}{2} < 1$, so $\sum a_n$ converges by the Ratio Test.

(c) $\sum \dfrac{1}{n\,(\ln n)^2}$

Solution: Let $a_n = \dfrac{1}{n(\ln n)^2}$, and let $f(x) = \dfrac{1}{x(\ln x)^2}$ so that $a_n = f(n)$. Note that $f(x)$ is decreasing for $x \geq 1$ and, setting $u = \ln x$ so $du = \dfrac{dx}{x}$, we have $\displaystyle\int_e^\infty f(x)\,dx = \int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \int_1^\infty \frac{1}{u^2}\,du = \Big[-\tfrac{1}{u}\Big]_1^\infty = 1$, and so $\sum a_n$ converges by the Integral Test.

(d) $\sum \dfrac{n^n}{n!}$

Solution: Let $a_n = \dfrac{n^n}{n!}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = \lim\limits_{n\to\infty} \dfrac{(n+1)^{n+1}}{(n+1)!} \dfrac{n!}{n^n} = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^n = e > 1$, so $\sum a_n$ diverges by the Ratio Test.

(e) $\sum \dfrac{\ln n}{\sqrt{n}}$

Solution: For $n \geq e$ we have $\ln n \geq 1$ so $\dfrac{\ln n}{\sqrt{n}} \geq \dfrac{1}{\sqrt{n}}$. But $\sum \dfrac{1}{\sqrt{n}}$ diverges, and so $\sum \dfrac{\ln n}{\sqrt{n}}$ diverges too, by the Comparison Test.

5: For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+2}$

Solution: Let $a_n = \dfrac{(-1)^n\sqrt{n}}{n+2}$. Then $|a_n| = \dfrac{\sqrt{n}}{n+2}$. Since the sequence $\{|a_n|\}$ decreases to 0, $\sum a_n$ converges by tha A.S.T. On the other hand, if we let $b_n = \dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}}$, then we have $\dfrac{|a_n|}{b_n} \longrightarrow 1$ and $\sum b_n$ diverges (it is a p-series with $p = \frac{1}{2}$), and so $\sum |a_n|$ diverges too, by the L.C.T. Thus $\sum a_n$ converges conditionally.

(b) $\sum(-1)^n e^{1/n}$

Solution: $\displaystyle\lim_{n\to\infty} e^{1/n} = 1$, since $\frac{1}{n} \to 0$, and so $\sum(-1)^n e^{1/n}$ diverges by the N^{th}-Term Test.

(c) $\displaystyle\sum \frac{(-1)^n}{\ln n}$

Solution: For $n > 1$, $\left\{\frac{1}{\ln n}\right\}$ is decreasing, and $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$, and so $\sum \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. On the other hand, $\frac{1}{\ln n} > \frac{1}{n}$, and $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{\ln n}$ diverges too, by the Comparison Test. Thus $\sum \frac{(-1)^n}{\ln n}$ is conditionally convergent.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$

Solution: Let $a_n = \dfrac{(-2)^n}{n!}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{2^{n+1}}{(n+1)!}\dfrac{n!}{2^n} = \dfrac{2}{n+1} \longrightarrow 0 < 1$, and so $\sum a_n$ converges absolutely by the R.T.

(e) $\displaystyle\sum \frac{n}{(-2)^n}$

Solution: Let $a_n = \dfrac{n}{(-2)^n}$. Then $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}}\frac{2^n}{n} = \frac{1}{2}$, so $\sum |a_n|$ converges by the Ratio Test. Thus $\displaystyle\sum \frac{n}{(-2)^n}$ is absolutely convergent.

6: (a) Estimate the sum $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n+2}$ so that the absolute error is at most $\frac{1}{30}$.

Solution: Let $a_n = \frac{(-1)^n}{2^n+2}$. Then $|a_n| = \frac{1}{2^n+2}$ so $\{|a_n|\}$ is decreasing with $\lim_{n\to\infty} |a_n| = 0$. By the Alternating Series Test, if we approximate $S = \sum_{n=0}^{\infty} a_n$ by the partial sum $S_l = \sum_{n=1}^{l} a_n$ then the absolute error is

$$E_l = \left|S - S_l\right| = \left|\sum_{n=l+1}^{\infty} a_n\right| \le |a_{l+1}| = \frac{1}{2^{l+1}+2}\,.$$

We have $\frac{1}{2^{l+1}+2} \le \frac{1}{30} \iff 2^{l+1}+2 \ge 30 \iff 2^{l+1} \ge 28 \iff l+1 \ge 5 \iff l \ge 4$, so to get $E_l \le \frac{1}{30}$ we can take $l = 4$. Finally, note that $S_4 = \sum_{n=0}^{4} \frac{(-1)^n}{2^n+2} = \frac{1}{3} - \frac{1}{4} + \frac{1}{6} - \frac{1}{10} + \frac{1}{18} = \frac{37}{180}$.

(b) Estimate the sum $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ so that the absolute error is at most $\frac{1}{16}$.

Solution: Let $a_n = \frac{1}{n^3+n}$. If we approximate $S = \sum_{n=1}^{\infty} a_n$ by the partial sum $S_l = \sum_{n=1}^{l} a_n$ then by the Comparison Test (since $\frac{1}{n^3+n} \le \frac{1}{n^3}$ for all $n \ge 1$) and the Integral Test (since $f(x) = \frac{1}{x^3}$ is decreasing for $x > 0$), the (absolute) error is

$$E_l = S - S_l = \sum_{n=l+1}^{\infty} \frac{1}{n^3+n} \le \sum_{n=l+1}^{\infty} \frac{1}{n^3} \le \int_l^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_l^{\infty} = \frac{1}{2l^2} - \lim_{x\to\infty} \frac{1}{2x^2} = \frac{1}{2l^2}\,.$$

We have $\frac{1}{2l^2} \le \frac{1}{16} \iff 2l^2 \ge 16 \iff l^2 \ge 8 \iff l \ge 3$, so to get $E_l \le \frac{1}{16}$ we can take $l = 3$. Finally, note that $S_3 = \sum_{n=1}^{\infty} \frac{1}{n^3+n} = \frac{1}{2} + \frac{1}{10} + \frac{1}{30} = \frac{19}{30}$.

(c) Estimate the sum $\sum_{n=2}^{\infty} \frac{n-1}{n!}$ so that the absolute error is at most $\frac{1}{100}$.

Solution: If we approximate $S = \sum_{n=2}^{\infty} \frac{n-1}{n!}$ by the partial sum $S_l = \sum_{n=2}^{l} \frac{n-1}{n!}$ then, by the Comparison Test, the (absolute) error is

$$\begin{aligned}
E_l = S - S_l &= \sum_{n=l+1}^{\infty} \frac{n-1}{n!} \\
&= \tfrac{l}{(l+1)!} + \tfrac{l+1}{(l+2)!} + \tfrac{l+2}{(l+3)!} + \tfrac{l+3}{(l+4)!} + \cdots \\
&= \tfrac{l}{l+1} \cdot \tfrac{1}{l!} + \tfrac{l+1}{l+2} \cdot \tfrac{1}{(l+1)!} + \tfrac{l+2}{l+3} \cdot \tfrac{1}{(l+2)!} + \tfrac{l+3}{l+4} \cdot \tfrac{1}{(l+3)!} + \cdots \\
&\le \tfrac{1}{l!} + \tfrac{1}{(l+1)!} + \tfrac{1}{(l+2)!} + \tfrac{1}{(l+3)!} + \cdots \\
&= \tfrac{1}{l!}\left(1 + \tfrac{1}{(l+1)} + \tfrac{1}{(l+1)(l+2)} + \tfrac{1}{(l+1)(l+2)(l+3)} + \cdots\right) \\
&\le \tfrac{1}{l!}\left(1 + \tfrac{1}{(l+1)} + \tfrac{1}{(l+1)^2} + \tfrac{1}{(l+1)^3} + \cdots\right) \\
&= \tfrac{1}{l!} \cdot \tfrac{1}{1-\frac{1}{l+1}} = \tfrac{1}{l!} \cdot \tfrac{l+1}{l}\,.
\end{aligned}$$

To get $E_l \le \frac{1}{100}$ we can choose l so that $\frac{l+1}{l\cdot l!} \le \frac{1}{100}$. By trial and error, we find that the smallest such value is $l = 5$. Finally note that $S_5 = \sum_{n=2}^{5} \frac{n-1}{n!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} = \frac{119}{120}$.

We remark that in fact $S_l = 1 - \frac{1}{l!}$ for all $l \ge 2$ so the exact value of the sum is $S = \lim_{l\to\infty} S_l = 1$.

7: Determine, with proof, which of the following statements are true.

(a) If $\sum a_n$ converges then $\sum \cos(a_n)$ diverges.

Solution: This is TRUE. Suppose that $\sum a_n$ converges. Then $a_n \to 0$ and so $\cos(a_n) \to 1$, and hence $\sum \cos(a_n)$ diverges by the NTT.

(b) If $a_n \geq 0$ for all n and $\sum a_n$ converges then $\sum {a_n}^2$ converges.

Solution: This is TRUE. Suppose that $a_n \geq 0$ for all n and $\sum a_n$ converges. Then $a_n \to 0$ and so for large values of n we have $a_n \leq 1$. But when $a_n \leq 1$ we have ${a_n}^2 \leq a_n$, and so $\sum {a_n}^2$ converges , by the CT.

(c) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$ then $\left(\sum a_n \text{ converges} \iff \sum b_n \text{ converges}\right)$.

Solution: This is false. For a counterexample, let $a_{2n} = \frac{1}{\sqrt{n}}$ and $a_{2n+1} = -\frac{1}{\sqrt{n}}$ for all $n \geq 1$, so we have $\{a_n\} = \left\{1, -1, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \cdots\right\}$, and let $b_{2n} = \left(1 + \frac{1}{\sqrt{n}}\right) a_{2n} = \left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right)$ and $b_{2n+1} = a_{2n+1} = -\frac{1}{\sqrt{n}}$. Note that $\sum a_n$ converges by the A.S.T. Also, we have $\dfrac{a_{2n+1}}{b_{2n+1}} = 1$ for all n and $\dfrac{a_{2n}}{b_{2n}} = \dfrac{1}{1 + \frac{1}{\sqrt{n}}} \to 1$ as $n \to \infty$, and so $\dfrac{a_n}{b_n} \to 1$ as $n \to \infty$. But $\sum b_n$ diverges, since, writing S_l for the l^{th} partial sum of $\sum\limits_{n=1}^{\infty} b_n$, we have $S_{2l+1} = \sum\limits_{n=1}^{l} (a_{2n} + a_{2n+1}) = \sum\limits_{n=1}^{l} \left(\left(\frac{1}{\sqrt{n}} + \frac{1}{n} \right) - \frac{1}{\sqrt{n}} \right) = \sum\limits_{n=1}^{l} \frac{1}{n} \to \infty$ as $l \to \infty$.

(d) If $f(x)$ is non-negative and continuous and $\displaystyle\int_1^\infty f(x)\,dx$ converges then $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges.

Solution: This is false, and we provide a counterexample. Let

$$g_1(x) = \begin{cases} 2x - 1 \text{ if } \frac{1}{2} \leq x \leq 1, \\ 3 - 2x \text{ if } 1 \leq x \leq \frac{3}{2}, \\ 0 \quad \text{otherwise}, \end{cases} \quad g_2(x) = \begin{cases} 4x - 7 \text{ if } \frac{7}{4} \leq x \leq 2, \\ 9 - 4x \text{ if } 2 \leq x \leq \frac{9}{4}, \\ 0 \quad \text{otherwise}, \end{cases} \quad g_3(x) = \begin{cases} 8x - 23 \text{ if } \frac{23}{8} \leq x \leq 3, \\ 25 - 8x \text{ if } 3 \leq x \leq \frac{25}{8}, \\ 0 \quad \text{otherwise}, \end{cases}$$

and in general, for $k \geq 1$ let

$$g_k(x) = \begin{cases} 2^k x - k2^k + 1 \text{ if } k - \frac{1}{2^k} \leq x \leq k, \\ k2^k + 1 - 2^k x \text{ if } k \leq x \leq k + \frac{1}{2^k}, \\ 0 \quad \text{otherwise}. \end{cases}$$

Then $\displaystyle\int_0^\infty g_1(x)\,dx = \frac{1}{2}$, $\displaystyle\int_0^\infty g_2(x)\,dx = \frac{1}{4}$, $\displaystyle\int_0^\infty g_3(x)\,dx = \frac{1}{8}$, and in general $\displaystyle\int_0^\infty g_k(x)\,dx = \frac{1}{2^k}$. Now let $g(x) = g_k(x)$ when $x \in \left[k - \frac{1}{2^k}, k + \frac{1}{2^k}\right]$ and let $g(x) = 0$ otherwise. The graph of $g(x)$ is shown below.

Then $g(x)$ is nonnegative and continuous, and $\displaystyle\int_1^\infty g(x)\,dx$ converges, indeed $\displaystyle\int_0^\infty g(x)\,dx = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$.

On the other hand we have $g(n) = 1$ for all integers $n \geq 1$, so $\displaystyle\sum_{n=1}^{\infty} g(n) = \infty$.