MATH 138 Calculus 2, Solutions to the Exercises for Chapter 8

1: (a) Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(1-4x)^n}{n\,2^n}$.

Solution: Let $a_n=\dfrac{(1-4x)^n}{n\,2^n}$. Then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|1-4x|^{n+1}}{(n+1)\,2^{n+1}}\,\frac{n\,2^n}{|1-4x|^n}=\lim_{n\to\infty}\tfrac{1}{2}|1-4x|\left(\tfrac{n}{n+1}\right)=\tfrac{1}{2}\left|1-4x\right|=2\left|x-\tfrac{1}{4}\right|$. By the Ratio Test, $\sum a_n$ converges when $\left|x-\frac{1}{4}\right|<\frac{1}{2}$ and diverges when $\left|x-\frac{1}{4}>\frac{1}{2}\right.$. Also, we have $\left|x-\frac{1}{4}\right|=\frac{1}{2}$ when $x=-\frac{1}{4}$ or $\frac{3}{4}$. When $x=-\frac{1}{4}$, $a_n=\frac{2^n}{n\,2^n}=\frac{1}{n}$ and $\sum a_n$ diverges, and when $x=\frac{3}{4}$, $a_n=\frac{(-2)^n}{n\,2^n}=\frac{(-1)^n}{n}$ and $\sum a_n$ converges. Thus the interval of convergence is $\left(-\frac{1}{4},\frac{3}{4}\right]$.

(b) Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(2-3x)^n}{n+\sqrt{n}}$.

Solution: Let $a_n=\dfrac{(2-3x)^n}{n+\sqrt{n}}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right|=\dfrac{|2-3x|^{n+1}}{n+1+\sqrt{n+1}}\cdot\dfrac{n+\sqrt{n}}{|2-3x|^n}=\dfrac{n+\sqrt{n}}{n+1+\sqrt{n+1}}\cdot|2-3x|$ and so $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|2-3x|$. By the Ratio Test, $\sum a_n$ converges when $|2-3x|<1$ and diverges when $|2-3x|>1$. We have $|2-3x|<1\iff -1<2-3x<1\iff 1>3x-2>-1\iff 3>3x>1\iff 1>x>\frac{1}{3}$. When $x=\frac{1}{3}$ we have $a_n=\frac{1}{n+\sqrt{n}}>\frac{1}{2n}$ and $\sum\frac{1}{2n}$ diverges (its a p-series) so $\sum a_n$ diverges too, by the Comparison Test. When $x=1$ we have $a_n=\dfrac{(-1)^n}{n+\sqrt{n}}$ so $\sum a_n$ converges by the Alternating Series Test, since $\left\{\frac{1}{n+\sqrt{n}}\right\}$ is decreasing with $\displaystyle\lim_{n\to\infty}\frac{1}{n+\sqrt{n}}=0$. Thus the interval of convergence is $I=\left(\frac{1}{3},1\right]$.

(c) Find the set of all values of x such that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(x^2+x-1)^n}{n}$ converges.

Solution: Let $a_n=\frac{(x^2+x-1)^n}{n}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right|=\dfrac{|x^2+x-1|^{n+1}}{n+1}\cdot\dfrac{n}{5|x^2+x-1|}=\dfrac{n+1}{n}|x^2+x-1|$ and so $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x^2+x-1|$. By the R.T, the series converges (absolutely) when $|x^2+x-1|<1$ and diverges when $|x^2+x-1|>1$. From the graph of $y=x^2+x-1$ we see that $|x^2+x-1|<1$ when $x\in(-2,-1)\cup(0,1)$ and $|x^2+x-1|>1$ when $x\in(-\infty,-2)\cup(-1,0)\cup(1,\infty)$. We check the endpoints of these intervals. When $x=-2,1$ we have $a_n=\frac{1}{n}$ so $\sum a_n$ converges, and when $x=-1,0$ $a_n=\frac{(-1)^n}{n}$ so $\sum a_n$ converges. Thus the series converges when $x\in(-2,-1]\cup[0,1)$ (note that the series is not a power series, and the set of convergence is not an interval).

2: (a) Find the Taylor Polynomial of degree 5 centred at 0 for $f(x) = \dfrac{e^x}{1+x}$.

Solution:
$$\begin{aligned} f(x) &= e^x \ \frac{1}{1+x} \\ &= \big(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 + \cdots\big)\big(1 - x + x^2 - x^3 + x^4 - x^5 - \cdots\big) \\ &= 1 + \big(-1+1\big)x + \big(1 - 1 + \tfrac{1}{2}\big)x^2 + \big(-1 + 1 - \tfrac{1}{2} + \tfrac{1}{6}\big)x^3 + \big(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24}\big)x^4 \\ &\quad + \big(-1 + 1 - \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{24} + \tfrac{1}{120}\big)x^5 + \cdots \\ &= 1 + 0x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{9}{24}x^4 - \tfrac{44}{120}x^5 + \cdots \end{aligned}$$
so the Taylor polynomial of degree 5 is $P_5(x) = 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{8}x^4 - \frac{11}{30}x^5$.

(b) Find the Taylor Polynomial of degree 5 centred at 0 for $f(x) = (1+2x)^{3/2}\sin x$.

Solution: $f(x) = (1+2x)^{3/2}\sin x$
$$\begin{aligned} &= \Big(1 + \tfrac{3}{2}(2x) + \tfrac{(\frac{3}{2})(\frac{1}{2})}{2!}(2x)^2 + \tfrac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{3!}(2x)^3 + \tfrac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{4!}(2x)^4 + \cdots\Big)\big(x - \tfrac{1}{6}x^3 + \tfrac{1}{120} \\ &= \big(1 + 3x + \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 + \tfrac{3}{8}x^4 + \cdots\big)\big(x + 0x^2 - \tfrac{1}{6}x^3 + 0x^4 + \tfrac{1}{120}x^5 + \cdots\big) \\ &= x + (0+3)x^2 + \big(-\tfrac{1}{6} + 0 + \tfrac{3}{2}\big)x^3 + \big(0 - \tfrac{1}{2} + 0 - \tfrac{1}{2}\big)x^4 + \big(\tfrac{1}{120} + 0 - \tfrac{1}{4} + 0 + \tfrac{3}{8}\big)x^5 + \cdots \\ &= x + 3x^2 + \tfrac{8}{6}x^3 - x^4 + \tfrac{16}{120}x^5 + \cdots \end{aligned}$$
so the Taylor polynomial of degree 5 is $P_5(x) = x + 3x^2 + \frac{4}{3}x^3 - x^4 + \frac{2}{15}x^5$.

(c) Find the Taylor polynomial of degree 4, centred at 0, for $f(x) = \dfrac{\ln(1+x)}{\tan^{-1} x}$.

Solution: We have $\dfrac{\ln(1+x)}{\tan^{-1}x} = \dfrac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots}{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots}$. We use long division:

$$\begin{array}{r|l} & 1 - \tfrac{1}{2}x + \tfrac{2}{3}x^2 - \tfrac{5}{12}x^3 + \tfrac{2}{9}x^4 + \cdots \\ \hline 1 - \tfrac{1}{3}x^2 + \tfrac{1}{5}x^4 - \cdots & 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \tfrac{1}{5}x^5 - \cdots \\ & 1 + 0\,x - \tfrac{1}{3}x^2 + 0\,x^3 + \tfrac{1}{5}x^4 + \cdots \\ \cline{2-2} & -\tfrac{1}{2}x + \tfrac{2}{3}x^2 - \tfrac{1}{4}x^3 + 0\,x^4 + \cdots \\ & -\tfrac{1}{2}x + 0\,x^2 + \tfrac{1}{6}x^3 + 0\,x^4 + \cdots \\ \cline{2-2} & \tfrac{2}{3}x^2 - \tfrac{5}{12}x^3 + 0\,x^4 + \cdots \\ & \tfrac{2}{3}x^2 + 0\,x^3 - \tfrac{2}{9}x^4 + \cdots \\ \cline{2-2} & -\tfrac{5}{12}x^3 + \tfrac{2}{9}x^4 + \cdots \end{array}$$

3: (a) Find the Taylor series centred at 0 for $f(x)=\dfrac{-4}{x^2+2x-3}$, and find the radius of convergence.

Solution: We have $f(x)=\dfrac{-4}{x^2+2x-3}=\dfrac{-1}{x-1}+\dfrac{1}{x+3}=\dfrac{1}{1-x}+\dfrac{\frac{1}{3}}{1+\frac{x}{3}}$. Since $\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$ and $\dfrac{1}{1+\frac{x}{3}}=\sum_{x=0}^{\infty}\left(-\frac{x}{3}\right)^n$ for $|x|<3$, $f(x)=\sum_{n=0}^{\infty}x^n+\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n x^n=\sum_{n=0}^{\infty}\left(1+\frac{1}{3}\left(-\frac{1}{3}\right)^n\right)x^n$ for $|x|<1$.

(b) Find the Taylor series centred at -1 for $f(x)=\dfrac{-4}{x^2+2x-3}$, and find the radius of convergence.

Solution: We have $f(x)=\dfrac{-1}{x-1}+\dfrac{1}{x+3}=\dfrac{-1}{(x+1)-2}+\dfrac{1}{(x+1)+2}=\dfrac{\frac{1}{2}}{1-\frac{x+1}{2}}+\dfrac{\frac{1}{2}}{1+\frac{x+1}{2}}$. Since we have $\dfrac{1}{1-\frac{x+1}{2}}=\sum_{n=0}^{\infty}\left(\frac{x+1}{2}\right)^n$ for $|x+1|<2$ and $\dfrac{1}{1+\frac{x+1}{2}}=\sum_{x=0}^{\infty}\left(-\frac{x+1}{2}\right)^n$ for $|x+1|<2$, we find that $f(x)=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x+1}{2}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x+1}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)(x+1)^n=\sum_{n=0}^{\infty}\dfrac{1}{2^{2n}}(x+1)^{2n}$ for $|x+|<2$.

(c) Find the Taylor series centred at 0 for $f(x)=\sin x\cos x$.

Solution: We have

$$f(x)=\sin x\cos x=\tfrac{1}{2}\sin(2x)=\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+2}(2n+1)!}\,x^{2n+1}\,.$$

(d) Find the Taylor series centred at $\frac{\pi}{4}$ for $f(x)=\sin x\cos x$.

Solution: We provide two solutions. The first solution uses the known Taylor series for $\cos x$. We have

$$\begin{aligned}f(x)&=\sin x\cos x=\tfrac{1}{2}\sin 2x=\tfrac{1}{2}\cos\left(2x-\tfrac{\pi}{2}\right)=\tfrac{1}{2}\cos\left(2\left(x-\tfrac{\pi}{4}\right)\right)\\&=\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(2\left(x-\tfrac{\pi}{4}\right)\right)^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n-1}}{(2n)!}\left(x-\tfrac{\pi}{4}\right)^{2n}.\end{aligned}$$

The second solution uses the formula for the coefficients of the Taylor series. We have $f(x)=\frac{1}{2}\sin 2x$, $f'(x)=\cos 2x$, $f''(x)=-2\sin 2x$, $f'''(x)=-4\cos 2x$, $f''''(x)=8\sin 2x$ and so on. Put in $x=\frac{\pi}{4}$ to get $f\left(\frac{\pi}{4}\right)=\frac{1}{2}$, $f'\left(\frac{\pi}{4}\right)=0$, $f''\left(\frac{\pi}{4}\right)=-2$, $f'''\left(\frac{\pi}{4}\right)=0$, $f''''\left(\frac{\pi}{4}\right)=8$ and so on. In general, the odd-order derivatives at 0 are all zero, that is $f^{(2n+1)}(0)=0$, and the even-order derivatives are given by $f^{(2n)}(0)=(-1)^n2^{2n-1}$. Thus the coefficients of the Taylor series are given by $c_{2n+1}=0$ and $c_{2n}=\dfrac{(-1)^n2^{2n-1}}{(2n)!}$, and so we have $f(x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n2^{2n-1}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}$.

4: (a) Approximate the value of $(1300)^{2/3}$ so that the absolute error is at most $\frac{1}{200}$.

Solution: We have

$$\begin{aligned}(1300)^{2/3} &= 100\left(1+\tfrac{3}{10}\right)^{2/3}\\ &= 100\Bigg(1+\left(\tfrac{2}{3}\right)\left(\tfrac{3}{10}\right)+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\left(\tfrac{3}{10}\right)^2+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}\left(\tfrac{3}{10}\right)^3\\ &\qquad +\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{4!}\left(\tfrac{3}{10}\right)^4+\cdots\Bigg)\\ &= 100+2\cdot 10-\tfrac{2\cdot 1}{2!}+\tfrac{2\cdot 1\cdot 4}{3!\,10}-\tfrac{2\cdot 1\cdot 4\cdot 7}{4!\,10^2}+\tfrac{2\cdot 1\cdot 4\cdot 7\cdot 10}{5!\,10^3}-\cdots\\ &\cong 100+20-1+\tfrac{4}{30}-\tfrac{7}{300}=119\tfrac{33}{300}=119.11\end{aligned}$$

with absolute error $E\le\frac{2\cdot 1\cdot 4\cdot 7\cdot 10}{5!\,10^3}=\frac{7}{1500}<\frac{1}{200}$, by the Alternating Series Test.

(b) Approximate $\ln(4/5)$ ensuring that the error is smaller than $\frac{1}{100}$.

Solution: Let $f(x)=\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\cdots$. Then $\ln\left(\frac{4}{5}\right)=f\left(\frac{1}{5}\right)=-\frac{1}{5}-\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}-\frac{1}{4\cdot 5^4}-\cdots\cong -\frac{1}{5}-\frac{1}{2\cdot 5^2}=-\frac{11}{50}$ and, using the Comparison Test, the absolute error is $E=\frac{1}{3\cdot 5^3}+\frac{1}{4\cdot 5^4}+\frac{1}{5\cdot 5^5}+\cdots<\frac{1}{3\cdot 5^3}+\frac{1}{3\cdot 5^4}+\frac{1}{3\cdot 5^5}+\cdots=\frac{\frac{1}{3\cdot 5^3}}{1-\frac{1}{5}}=\frac{1}{300}<\frac{1}{100}$.

(c) Approximate $\displaystyle\int_0^{1/5}\frac{\ln(1+x)}{x}\,dx$ so that the error is less than $\frac{1}{1000}$.

Solution: $\displaystyle\int_0^{1/5}\frac{\ln(1+x)}{x}\,dx=\int_0^{1/5}\frac{x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots}{x}\,dx=\int_0^{1/5}1-\tfrac{1}{2}x+\tfrac{1}{3}x^2-\tfrac{1}{4}x^3+\cdots\,dx=$ $\left[x-\frac{1}{2^2}x^2+\frac{1}{3^2}x^3-\frac{1}{4^2}x^4+\cdots\right]_0^{1/5}=\frac{1}{5}-\frac{1}{2^25^2}+\frac{1}{3^25^3}+\frac{1}{4^25^4}+\cdots\cong\frac{1}{5}-\frac{1}{2^25^2}=\frac{19}{100}$ with error $E<\frac{1}{3^25^3}=\frac{1}{1125}<\frac{1}{1000}$, by the Alternating Series Test.

5: (a) Let $f(x)=\cos^2\left(\dfrac{x^2}{4\sqrt{3}}\right)$. Find the twelfth derivative $f^{(12)}(0)$.

Solution: Since $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$, we have

$$\begin{aligned} f(x)=\cos^2\left(\frac{x^2}{4\sqrt{3}}\right)&=\tfrac{1}{2}+\tfrac{1}{2}\cos\left(\frac{x^2}{2\sqrt{3}}\right)=\tfrac{1}{2}+\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{x^2}{2\sqrt{3}}\right)^{2n}\\ &=1-\frac{1}{2!\,2^3\,3^1}\,x^4+\frac{1}{4!\,2^5\,3^2}\,x^8-\frac{1}{6!\,2^7\,3^3}\,x^{12}+\cdots, \end{aligned}$$

and so $f^{(12)}(0)=12!\,c_{12}=\dfrac{12!}{6!\,2^7\,3^3}=-\dfrac{11\cdot 5\cdot 7}{2}=-\frac{385}{2}$.

(b) Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin x\tan^{-1}x-x^2}{\cos(x^2)-1}$.

Solution: $\displaystyle\lim_{x\to 0}\frac{\sin x\tan^{-1}x-x^2}{\cos(x^2)-1}=\lim_{x\to 0}\frac{\left(x-\frac{1}{6}x^3+\cdots\right)\left(x-\frac{1}{3}x^3+\cdots\right)-x^2}{\left(1-\frac{1}{2}x^4+\cdots\right)-1}=\lim_{x\to 0}\frac{-\frac{1}{2}x^4+\cdots}{-\frac{1}{2}x^4+\cdots}=1.$

(c) Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}$.

Solution: Recall that $\displaystyle\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,x^{2n}$, so $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,(\sqrt{2})^{2n}=\cos(\sqrt{2})$.

(d) Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{n}{(n+1)!}$.

Solution: We have $\displaystyle e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}$ so $\displaystyle\frac{e^x-1}{x}=\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}$. Differentiate both sides to get $\displaystyle\frac{x\,e^x-(e^x-1)}{x^2}=\sum_{n=0}^{\infty}\frac{n\,x^{n-1}}{(n+1)!}$. Then put in $x=1$ to get $\displaystyle\sum_{n=0}^{\infty}\frac{n}{(n+1)!}=1$.

5: (a) Let $c_n = 1$ when n is even and $c_n = 2$ when n is odd. Find the function $f(x)$ whose Taylor series centered at 0 is equal to $\sum_{n=0}^{\infty} c_n x^n$.

Solution: We have

$$\begin{aligned}\sum_{n=0}^{\infty} c_n x^n &= 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots \\ &= \big(1 + x^2 + x^4 + x^6 + \cdots\big) + \big(2x + 2x^3 + 2x^5 + 2x^7 + \cdots\big) \\ &= \frac{1}{1-x^2} + \frac{2x}{1-x^2} = \frac{1+2x}{1-x^2}\end{aligned}$$

when $|x| < 1$, and so $f(x) = \dfrac{1+2x}{1-x^2}$.

(b) Let $f(x) = x^3 + x + 1$. Note that $f(x)$ is increasing with $f(0) = 1$. Let $g(x) = f^{-1}(x)$, Find the Taylor polynomial of degree 6 centered at 1 for $g(x)$.

Solution: Say $g(y) = a_0 + a_1(y-1) + a_2(y-1)^2 + a_3(y-1)^3 + \cdots$. Then

$$\begin{aligned}x = g\big(f(x)\big) &= g(x^3 + x + 1) = a_0 + a_1(x + x^3) + a_2(x + x^3)^2 + a_3(x + x^3)^3 + \cdots \\ &= a_0 + a_1(x + x^3) + a_2(x^2 + 2x^4 + x^6) + a_3\big(x^3 + 3x^5 + \cdots\big) \\ &\qquad + a_4\big(x^4 + 4x^6 + \cdots\big) + a_5\big(x^5 + \cdots\big) + a_6\big(x^6 + \cdots\big) + \cdots \\ &= a_0 + a_1 x + a_2 x^2 + (a_3 + a_1)x^3 + (a_4 + 2a_2)x^4 + (a_5 + 3a_3)x^5 + (a_6 + 4a_4 + a_2)x^6 + \cdots\end{aligned}$$

Comparing coefficients, we see that $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -a_1 = -1$, $a_4 = -2a_2 = 0$, $a_5 = -3a_3 = 3$ and $a_6 = -4a_4 - a_2 = 0$, and so the 6^{th} Taylor polynomial is $T_6(x) = (x-1) - (x-1)^3 + 3(x-1)^5$.

(c) Find the Taylor polynomial of degree 5 centred at 0 for the solution to the IVP $\frac{1}{2}\,y'' + y' - 3y = x + 1$ with $y(0) = 1$ and $y'(0) = 2$.

Solution: Let $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$. Then $y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots$ and $y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots$. So y is a solution to the DE when

$$\begin{aligned}0 &= \tfrac{1}{2}\,y'' + y' - 3y - x - 1 \\ &= (c_2 + 3c_3 x + 6c_4 x^2 + 10c_5 x^3 + \cdots) + (c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots) \\ &\qquad - (3c_0 + 3c_1 x + 3c_2 x^2 + 3c_3 x^3 + \cdots) - x - 1 \\ &= (c_2 + c_1 - 3c_0 - 1) + (3c_3 + 2c_2 - 3c_1 - 1)x + (6c_4 + 3c_3 - 3c_2)x^2 + (10c_5 + 4c_4 - 3c_3)x^3 + \cdots\end{aligned}$$

Since $y(0) = 1$ and $y'(0) = 2$ we have $c_0 = 1$ and $c_2 = 2$. Put these values in the above equation to get

$$0 = (c_2 - 2) + (3c_3 + 2c_2 - 7)x + (6c_4 + 3c_3 - 3c_2)x^2 + (10c_5 + 4c_4 - 3c_3)x^3 + \cdots$$

For y to be a solution, all the coefficients must be zero, so we have

$$\begin{aligned}&(c_2 - 2) = 0 \Longrightarrow c_2 = 2 \\ &(3c_3 + 2c_2 - 7) = 0 \Longrightarrow 3c_3 = 7 - 2c_2 = 3 \Longrightarrow c_3 = 1 \\ &(6c_4 + 3c_3 - 3c_2) = 0 \Longrightarrow 6c_4 = 3c_2 - 3c_3 = 3 \Longrightarrow c_4 = \tfrac{1}{2} \\ &(10c_5 + 4c_4 - 3c_3) = 0 \Longrightarrow 10c_5 = 3c_3 - 4c_4 = 1 \Longrightarrow c_5 = \tfrac{1}{10}\,.\end{aligned}$$

Thus the Taylor polynomial of degree 5 centered at 0 is

$$T_5(x) = 1 + 2x + 2x^2 + x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{10}x^5\,.$$