MATH 138 Calculus 2, Solutions to the Midterm Test, Winter 2024

[10] **1:** (a) Recall that $\sum\limits_{k=1}^{n} 1 = n$, $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.
Find $\displaystyle\int_{-1}^{2} x^2+2x\,dx$ by finding the limit off a sequence of Riemann sums for $f(x)=x^2+2x$.

Solution: We have

$$\begin{aligned}\int_{-1}^{2} x^2+2x\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f\big(-1+\tfrac{3k}{n}\big)\cdot\tfrac{3}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\Big(\big(1-\tfrac{6k}{n}+\tfrac{9k^2}{n^2}\big)+2\big(-1+\tfrac{3k}{n}\big)\Big)\cdot\tfrac{3}{n}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\big(-\tfrac{3}{n}+\tfrac{27k^2}{n^3}\big) = \lim_{n\to\infty}\Big(-\tfrac{3}{n}\sum_{k=1}^{n}1+\tfrac{27}{n^3}\sum_{k=1}^{n}k^2\Big)\\ &= \lim_{n\to\infty}\big(-\tfrac{3}{n}\cdot n+\tfrac{27}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6}\big) = -3+\tfrac{27\cdot 2}{6} = 6.\end{aligned}$$

(b) Approximate $\displaystyle\int_{-1}^{1}\frac{dx}{2x^2+x+3}$ using the Trapezoid Rule T_4 (on 4 sub-intervals).

Solution: Let $f(x)=\frac{1}{2x^2+x+3}$. When $[a,b]=[-1,1]$ is partitioned into $n=4$ equal-sized subintervals, the endpoints are $x_0=-1$, $x_1=-\frac{1}{2}$, $x_3=0$, $x_4=\frac{1}{2}$ and $x_5=1$, so we have

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \cong T_4 &= \tfrac{b-a}{2n}\big(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\big)\\ &= \tfrac{1}{4}\big(f(-1)+2f(-\tfrac{1}{2})+2f(0)+2f(\tfrac{1}{2})+f(1)\big)\\ &= \tfrac{1}{4}\big(\tfrac{1}{4}+2\cdot\tfrac{1}{3}+2\cdot\tfrac{1}{3}+2\cdot\tfrac{1}{4}+\tfrac{1}{6}\big) = \tfrac{1}{4}\cdot\tfrac{3+8+8+6+2}{12} = \tfrac{1}{4}\cdot\tfrac{27}{12} = \tfrac{9}{16}.\end{aligned}$$

[10] **2:** (a) Find $\displaystyle\int_{0}^{4}\frac{x\,dx}{\sqrt{2x+1}}$.

Solution: Let $u=2x+1$ so $du=2\,dx$. Then

$$\begin{aligned}\int_{x=0}^{4}\frac{x\,dx}{\sqrt{2x+1}} &= \int_{u=1}^{9}\frac{\frac{u-1}{2}\,\frac{1}{2}\,du}{\sqrt{u}} = \int_{u=1}^{9}\tfrac{1}{4}u^{1/2}-\,-\tfrac{1}{4}u^{-1/2}\,du\\ &= \Big[\tfrac{1}{6}u^{3/2}-\tfrac{1}{2}u^{1/2}\Big]_{u=1}^{9} = \big(\tfrac{9}{2}-\tfrac{3}{2}\big)-\big(\tfrac{1}{6}-\tfrac{1}{2}\big) = \tfrac{10}{3}.\end{aligned}$$

(b) Find $\displaystyle\int_{1}^{4}\frac{\ln x}{x^{3/2}}\,dx$.

Solution: Integrate by parts using $u=\ln x$, $du=\frac{1}{x}\,dx$, $v=-2x^{-1/2}$ and $dv=x^{-3/2}$ to get

$$\begin{aligned}\int_{1}^{4}\frac{\ln x}{x^{3/2}}\,dx &= \Big[-2x^{-1/2}\ln x+\int 2x^{-3/2}\,dx\Big]_{1}^{4} = \Big[-2x^{-1/2}\ln x-4x^{-1/2}\Big]_{1}^{4}\\ &= \big(-\ln 4-2\big)-\big(-4\big) = 2-2\ln 2.\end{aligned}$$

(c) Find $\displaystyle\int_{0}^{\pi/2}\frac{\cos^3 x}{2+\sin x}\,dx$.

Solution: Let $u=2+\sin x$ so $du=\cos x\,dx$. Then

$$\begin{aligned}\int_{0}^{\pi/2}\frac{\cos^3 x}{2+\sin x}\,dx &= \int_{0}^{\pi/2}\frac{1-\sin^2 x}{2+\sin x}\,\cos x\,dx = \int_{u=2}^{3}\frac{1-(u-2)^2}{u}\,du = \int_{u=2}^{3}\frac{-u^2+4u-3}{u}\,du\\ &= \int_{u=2}^{3}-u+4-\tfrac{3}{u}\,du = \Big[-\tfrac{1}{2}u^2+4u-3\ln u\Big]_{u=2}^{3}\\ &= \big(-\tfrac{9}{2}+12-3\ln 3\big)-\big(-2+8-3\ln 2\big) = \tfrac{3}{2}-3\ln\tfrac{3}{2}.\end{aligned}$$

[10] **3:** (a) Find $\displaystyle\int_0^\infty (2x+3)\,e^{-2x}\,dx$.

Solution: Integrate by parts using $u=2x+3$, $du=2\,dx$, $v=-\frac{1}{2}e^{-2x}$ and $dv=e^{-2x}\,dx$ to get

$$\begin{aligned}\int_0^\infty (2x+3)\,e^{-2x}\,dx &= \Big[-\tfrac{1}{2}(2x+3)e^{-2x}+\int e^{-2x}\,dx\Big]_0^\infty = \Big[-\tfrac{1}{2}(2x+3)e^{-2x}-\tfrac{1}{2}e^{-2x}\Big]_0^\infty \\ &= \Big[-\big(x+2\big)\,e^{-2x}\Big]_0^\infty = 2\end{aligned}$$

since $\lim\limits_{x\to\infty}(x-2)e^{-2x}=0$.

(b) Find $\displaystyle\int_0^3 \frac{dx}{\sqrt{x}\sqrt{3-x}}$.

Solution: First let $u=\sqrt{x}$ so $u^2=x$ and $2u\,du=dx$, then let $\sqrt{3}\,\sin\theta=u$ so that $\sqrt{3}\,\cos\theta=\sqrt{3-u^2}$ and $\sqrt{3}\cos\theta\,d\theta=du$, to get

$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt{x}\sqrt{3-x}} &= \int_{u=0}^{\sqrt{3}} \frac{2u\,du}{u\sqrt{3-u^2}} = \int_{u=0}^{\sqrt{3}} \frac{2\,du}{\sqrt{3-u^2}} = \int_{\theta=0}^{\pi/2} \frac{2\sqrt{3}\,\cos\theta\;d\theta}{\sqrt{3}\,\cos\theta} \\ &= \int_{\theta=0}^{\pi/2} 2\,d\theta = \Big[2\theta\Big]_{\theta=0}^{\frac{\pi}{2}} = \pi\,.\end{aligned}$$

We remark that the above integral is improper at both endpoints.

(c) Find $\displaystyle\int_0^\infty \frac{2x-1}{(x+1)(x^2+2x+2)}\,dx$.

Solution: To get $\frac{4x-2}{(x+1)(x^2+2x+2)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}$ for all x, we need $A(x^2+2x+2)+(Bx+C)(x+1)=2x-1$. Equate coefficients to get $A+B=0$ (1), $2A+B+C=2$ (2) and $2A+C=-1$, and put in $x=-1$ to get $A=-3$. Solve these to get $A=-3$, $B=3$ and $C=5$. Thus, noting that $x^2+2x+2=(x+1)^2+1$,

$$\begin{aligned}\int_0^\infty \frac{2x-1}{(x+1)(x^2+2x+2)}\,dx &= \int_0^\infty \frac{-3}{x+1}+\frac{3x+5}{x^2+2x+2}\,dx = \int -\frac{3}{x+1}+\frac{3x+3}{x^2+2x+2}+\frac{2}{(x+1)^2+1}\,dx \\ &= \Big[-3\ln(x+1)+\tfrac{3}{2}\ln(x^2+2x+2)+2\tan^{-1}(x+1)\Big]_0^\infty = \Big[\tfrac{3}{2}\ln\tfrac{x^2+2x+2}{(x+1)^2}+2\tan^{-1}(x+1)\Big]_0^\infty \\ &= \big(0+\pi\big)-\big(\tfrac{3}{2}\ln 2-\tfrac{\pi}{2}\big)=\tfrac{\pi}{2}-\tfrac{3}{2}\ln 2\,.\end{aligned}$$

[10] **4:** (a) Find $h'(1)$ when $h(x) = \displaystyle\int_{t=\sqrt{x}}^{\ln(x^3)} 4^{t^2}\,dt$.

Solution: Let $f(t) = 4^{t^2}$, let $g(y) = \int_{t=0}^{y} f(t)\,dt$, let $u(x) = \sqrt{x}$ and let $v(x) = \ln(x^3) = 3\ln x$. By the FTC we have $g'(y) = f(y) = 4^{y^2}$, and by decomposition we have

$$h(x) = \int_{t=u(x)}^{v(x)} f(t)\,dt = \int_{t=0}^{v(t)} f(t)\,dt - \int_{t=0}^{u(x)} f(t)\,dt = g\big(v(x)\big) - g\big(u(x)\big)$$

so by the Chain Rule, we have

$$h'(x) = g'(u(x))u'(x) - g'(v(x))v'(x) = 4^{u(x)^2}u'(x) - 4^{v(x)^2}v'(x) = 4^{(3\ln x)^2}\cdot\tfrac{3}{x} - 4^x\cdot\tfrac{1}{2\sqrt{x}}\,.$$

In particular, $h'(1) = 4^0\cdot 3 - 4^1\cdot\frac{1}{2} = 3 - 2 = 1$.

(b) Find $\displaystyle\int e^x \sin^2 x\,dx$.

Solution: We give two solutions. For the first solution, we let $I = \int \sin^2 x\,e^x\,dx$ and integrate by parts twice, first using $u = \sin^2 x$ so $du = 2\sin x\cos x\,dx$ and $v = e^x$ so $dv = e^x\,dx$, and then using $u = 2\sin x\cos x$ so $du = 2(\cos^2 x - \sin^2 x)\,dx = 2(1 - 2\sin^2 x)\,dx$ and $v = e^x$ do $dv = e^x\,dx$ to get

$$\begin{aligned} I &= \int \sin^2 x\ e^x\ dx = \sin^2 x\ e^x - \int 2\sin x\cos x\ e^x\,dx \\ &= \sin^2 x\ e^x - 2\sin x\cos x\ e^x + \int 2(1 - 2\sin^2 x)\,e^x\,dx \\ &= \sin^2 x\ e^x - 2\sin x\cos x\ e^x + 2e^x - 4I \end{aligned}$$

so that $5I = (\sin^2 x - 2\sin x\cos x + 2)\,e^x$ (plus an arbitrary constant), hence

$$\int \sin^2 x\,e^x\ dx = I = \tfrac{1}{5}(\sin^2 x - 2\sin x\cos x + 2)\,e^x\,.$$

For the second solution, we let $J = \int \cos 2x\,e^x\,dx$ and integrate twice, first using $u = \cos 2x$ so $du = -2\sin 2x$ and $v = e^x$ so $dv = e^x\,dx$, then using $u = 2\sin 2x$ so $du = 4\cos 2x$ and $v = e^x$ so $dv = e^x\,dx$, to get

$$\begin{aligned} J &= \int \cos 2x\ e^x\ dx = \cos 2x\ e^x n + \int 2\sin 2x\ e^x\ dx \\ &= \cos 2x\ e^x + 2\sin 2x\ e^x - \int 4\cos 2x\ e^x\ dx \\ &= \cos 2x + 2\sin 2x\ e^x - 4I \end{aligned}$$

so that $5I = (\cos 2x + 2\sin 2x)e^x$ (plus an arbitrary constant), hence

$$\int \cos 2x\ e^x\ dx = J = \tfrac{1}{5}(\cos 2x + 2\sin 2x).$$

Thus

$$\begin{aligned} \int \sin^2 x\ e^x\ dx &= \int \tfrac{1}{2}(1 - \cos 2x)\,e^x\ dx = \tfrac{1}{2}e^x - \tfrac{1}{2}J \\ &= \tfrac{1}{2}e^x - \tfrac{1}{10}(\cos 2x + 2\sin 2x)\,e^x \\ &= \tfrac{1}{10}\big(5 - \cos 2x - 2\sin 2x\big)\,e^x\,. \end{aligned}$$