MATH 247 Calculus 3, Solutions to Assignment 1

1: Consider the surface given implicitly by $x^2 + y^2 + 1 = yz$.

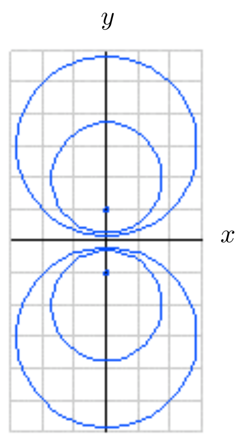
(a) Sketch the level sets $z = \pm 2, \pm 4, \pm 6$ and the level set $x = 0$ for this surface.

Solution: The level set $z = c$ is given by $x^2 + y^2 + 1 = cy$, or equivalently $x^2 + \left(y - \frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2 - 1$. For $c = \pm 2$ this becomes $x^2 + \left(y - \frac{c}{2}\right)^2 = 0$, so the level set is the single point $(x, y) = \left(0, \frac{c}{2}\right)$. For $c = \pm 4, \pm 6$, the level set is the circle centered at $\left(0, \frac{c}{2}\right)$ of radius $r = \sqrt{\left(\frac{c}{2}\right)^2 - 1}$. We make a table showing the values of r then plot the level sets.

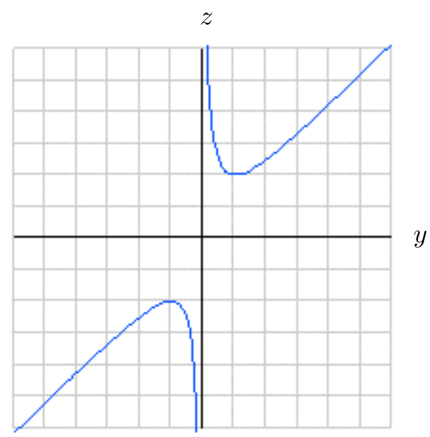
The level set $x = 0$ is given by $y^2 + 1 = yz$, that is $z = y + \frac{1}{y}$. This is a hyperbola with vertical asymptote along the z-axis and a slant asymptote along $z = y$.

c	r
± 2	0
± 4	$\sqrt{3} \cong 1.7$
± 6	$\sqrt{8} \cong 2.8$

Level Sets $z = \pm 2, \pm 4, \pm 6$

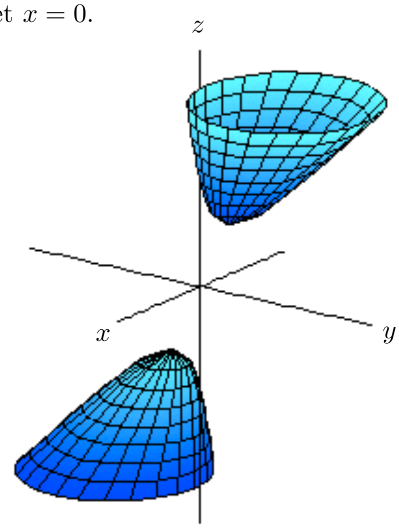


Level Set $x = 0$



(b) Sketch the surface.

Solution: To sketch the surface, we raise the level sets $z = c$ to the appropriate height and fit them into an envelope in the shape of the level set $x = 0$.



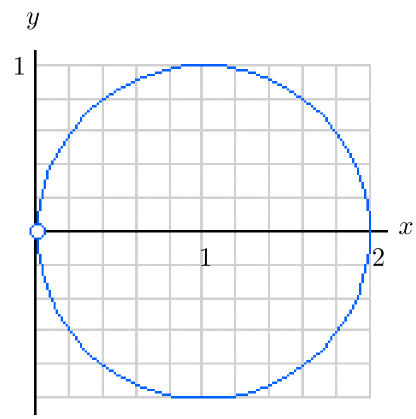
(c) Find an explicit equation for the tangent plane to the surface at the point $(2, 1, 6)$.

Solution: Let $g(x, y, z) = x^2 + y^2 - yz - 1$ so the surface is $\text{Null}(g)$. We have $Dg(x, y, z) = \left(2x, 2y - z, -y\right)$ so that $Dg(2, 1, 6) = (4, -4, -1)$. The tangent plane at $(2, 1, 6)$ is given by $4(x - 2) - 4(y - 1) - (z - 6) = 0$, or explicitly by $z = 4x - 4y + 2$.

2: (a) Sketch the curve given parametrically by $(x,y) = \left(\frac{2}{1+t^2}, \frac{2t}{1+t^2}\right)$, showing all points at which the tangent line is horizontal or vertical, then find an implicit equation for the curve.

Solution: We have $x'(t) = \dfrac{-4t}{(1+t^2)^2}$ and $y'(t) = \dfrac{2(1+t^2)-(2t)(2t)}{(1+t^2)^2} = \dfrac{2-2t^2}{(1+t^2)^2}$. The curve is horizontal when $y'(t) = 0$, that is when $t = \pm 1$, and vertical when $x'(t) = 0$, that is at $t = 0$. We make a table of values and sketch the curve.

t	x	y
$\to -\infty$	0	0
-3	$1/5$	$-3/5$
-2	$2/5$	$-4/5$
-1	1	-1
$-\frac{1}{2}$	$8/5$	$-4/5$
0	2	0
$\frac{1}{2}$	$8/5$	$4/5$
1	1	1
2	$2/5$	$4/5$
3	$1/5$	$3/5$
$\to \infty$	0	0



From the sketch, it appears that the curve is the circle of radius 1 centred at $(1,0)$ (with the origin removed), which has equation $(x-1)^2 + y^2 = 1$, or equivalently $x^2 + y^2 = 2x$. Let us verify that this is indeed the case. Suppose that (x,y) is on the curve, say $(x,y) = \left(\frac{2}{1+y^2}, \frac{2t}{1+y^2}\right)$. Then $x^2 = \frac{4}{(1+t^2)^2}$ and $y^2 = \frac{4t^2}{(1+t^2)^2}$ so that $x^2 + y^2 = \frac{4+4t}{(1+t^2)^2} = \frac{4}{1+t^2} = 2x$. Suppose, conversely, that (x,y) satisfies the equation $x^2 + y^2 = 2x$ with $x \neq 0$. Then for $\frac{t=y}{x}$ we have $\frac{2}{1+t^2} = \frac{2}{1+\left(\frac{y}{x}\right)^2} = \frac{2x^2}{x^2+y^2} = \frac{2x^2}{2x} = x$ and $\frac{2t}{1+t^2} = \frac{2}{1+t^2}\cdot t = x \cdot \frac{y}{x} = y$ so that (x,y) is on the curve. Thus the curve is given implicitly by $x^2 + y^2 = 2x$.

(b) Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = \left(t^2, \frac{t}{t^2+1}\right)$ and define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = y^2(x+1)^2 - x$.
Prove that $\text{Range}(f) = \text{Null}(g)$, then find an explicit equation for the tangent line to this curve at $\left(\frac{1}{4}, \frac{2}{5}\right)$.

Solution: Suppose $(x,y) \in \text{Range}(f)$. Choose $t \in \mathbb{R}$ so that $(x,y) = f(t) = \left(t^2, \frac{t}{t^2+1}\right)$, that is $x = t^2$ and $y = \frac{t}{t^2+1}$. Then $g(x,y) = y^2(x+1)^2 - x = \left(\frac{t}{t^2+1}\right)^2(t^2+1)^2 - t^2 = 0$, and so $(x,y) \in \text{Null}(g)$.

Suppose that $(x,y) \in \text{Null}(g)$ so that we have $0 = g(x,y) = y^2(x^2+1)^2 - x$, that is $y^2(x+1)^2 = x$. Let $t = y(x+1)$. Then we have $t^2 = y^2(x+1)^2 = x$ and $\frac{t}{t^2+1} = \frac{y(x+1)}{x^2+1} = y$, so that $(x,y) = f(t)$, and hence $(x,y) \in \text{Range}(f)$. We remark that, rather than choosing $t = y(x+1)$, we could have chosen $t = \sqrt{x}$ when $y \geq 0$ and $t = -\sqrt{x}$ when $y \leq 0$.

We can find the tangent line either by using either the formula for f or the formula for g. Let us use the formula for g. We have $g(x,y) = x^2y^2 + 2xy^2 + y^2 - x$ so that $Dg(x,y) = \left(2xy^2 + 2y^2 - 1, 2x^2y + 4xy + 2y\right)$ and hence $Dg\left(\frac{1}{4}, \frac{2}{5}\right) = \left(\frac{2}{25} + \frac{8}{25} - 1, \frac{1}{20} + \frac{2}{5} + \frac{4}{5}\right) = \left(-\frac{3}{5}, \frac{5}{4}\right)$. The tangent line is given by $-\frac{3}{5}\left(x - \frac{1}{4}\right) + \frac{5}{4}\left(y - \frac{2}{5}\right) = 0$, that is $-12x + 25y = -7$, or explicitly by $y = \frac{12}{25}x + \frac{7}{25}$.

3: (a) Find a parametric equation for the tangent line to the curve of intersection of the paraboloid $z = 1-x^2-y^2$ with the plane $z = 1-2x$ at the point $(1,1,-1)$.

Solution: We provide two solutions. For the first solution, note that the paraboloid is given explicitly by $z = f(x,y)$ where $f(x,y) = 1-x^2-y^2$ and we have $f(1,1) = -1$. The derivative matrix is $Df(x,y) = \big(-2x, -2y\big)$, so that $Df(1,1) = (-2,-2)$, and so the tangent plane at $(1,1,-1)$ is given explicitly by

$$\begin{aligned} z &= f(1,1) + Df(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = -1 + \big(-2,-2\big)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\ &= -1-2(x-1)-2(y-1) = -2x-2y+3. \end{aligned}$$

The plane $z = 1-2x$ is, of course, equal to its own tangent plane. The tangent line to the given curve C is the line of intersection of these two tangent planes, so we solve the two equations $z = -2x-2y+3$ (1) and $z = 1-2x$ (2). We let $x = t$, then equation (1) gives $z = 1-2t$ and equation (2) gives $2y = 3-2x-z = 3-2t-1+2t = 2$ so that $y = 1$. Thus the tangent line is given parametrically by

$$(x,y,z) = (t,1,1-2t) = (0,1,1) + t(1,0,-2).$$

For the second solution, we find a parametric equation for the curve C. We have $(x,y,z) \in C$ when $z = 1-x^2-y^2$ and $z = 1-2x$, that is when $1-x^2-y^2 = 1-2x$ and $z = 1-2x$. and we have $1-x^2-y^2 = 1-2x \iff x^2-2x+y^2 = 0 \iff (x-1)^2+y^2 = 1$. Thus $(x,y,z) \in C$ if and only if (x,y) lies on the circle $(x-1)^2+y^2 = 1$ and $z = 1-2t$. The circle $(x-1)^2+y^2 = 1$ is given parametrically by $(x,y) = \big(1+\sin t, \cos t\big)$ and we need $z = 1-2x = 1-2(1+\sin t) = -1-2\sin t$, and so the curve C is given by

$$(x,y,z) = \alpha(t) = \big(1+\sin t, \cos t, -1-2\sin t\big)$$

with $\alpha(0) = (1,1,-1)$. We have $\alpha'(t) = \big(\cos t, -\sin t, -2\cos t\big)$ so that $\alpha'(0) = (1,0,-2)$ and so the tangent line to the curve C at $(1,1,-1)$ is given by

$$(x,y,z) = \alpha(0) + t\,\alpha'(0) = (1,1,-1) + t\,(1,0,-2).$$

(b) When we consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, it is given by $f(x,y) = \big(u(x,y), v(x,y)\big)$ with $u(x,y) = x^2-y^2$ and $v(x,y) = 2xy$. Let $A = \big\{(x,y)\in\mathbb{R}^2 \,\big|\, 1\le x\le 2, 0\le y\le x\big\}$ and $B = \big\{(u,v)\in\mathbb{R}^2 \,\big|\, 1\le u\le 4\,,\, 0\le v\le 2\big\}$. Accurately sketch or describe the sets $f(A)$ and $f^{-1}(B)$.

Solution: Let us find the image of A. Consider the boundary. The line segment in the xy-plane from $(1,0)$ to $(2,0)$ is given by $(x,y) = \alpha(t) = (t,0)$ with $1 \le t \le 2$, and it is mapped by f to the curve $(u,v) = f(\alpha(t)) = f(t,0) = (t^2,0)$ with $1 \le t \le 2$, which is the line segment in the uv-plane from $(1,0)$ to $(4,0)$. The line segment in the xy-plane from $(1,1)$ to $(2,2)$ is given by $(x,y) = \beta(t) = (t,t)$ with $1 \le t \le 2$, and it is mapped by f to the curve $(u,v) = f(\beta(t)) = f(t,t) = (0,2t^2)$ with $1 \le t \le 2$, which is the line segment in the uv-plane from $(0,2)$ to $(0,8)$. When $a > 0$, the line segment in the xy-plane from $(a,0)$ to (a,a) is given by $(x,y) = \gamma_a(t) = (a,t)$ with $0 \le t \le a$, and it is mapped by f to the curve $(u,v) = f(\gamma_a(t)) = f(a,t) = (a^2-t^2, 2at)$ with $0 \le t \le a$. When $(u,v) = (a^2-t^2,2at)$, we have $v^2 = 4a^2t^2 = 4a^2(a^2-u)$ so that (u,v) lies on the parabola $u = a^2 - \frac{1}{4a^2}v^2$, which is the parabola in the uv-plane with vertex at $(a^2,0)$ and opens to the left passing through the points $(0,\pm 2a^2)$: the curve $(u,v) = f(\gamma_a(t)) = (a^2-t^2,2at)$ with $0 \le t \le a$ follows this parabola from $(a^2,0)$ to $(0,2a^2)$. Thus $f(A)$ is the region in the first quadrant of the uv-plane between two parabolas given by $u \ge 0$, $v \ge 0$, $1-\frac{1}{4}v^2 \le u \le 4-\frac{1}{16}v^2$.

Let us find $f^{-1}(B)$. For $(u,v) = f(x,y) = (x^2-y^2,2xy)$, we have $1 \le u \le 4 \iff 1 \le x^2-y^2 \le 4$ and we have $0 \le x \le 2 \iff 0 \le 2xy \le 2 \iff 0 \le xy \le 1$. The curves $x^2-y^2 = 1$ and $x^2-y^2 = 4$ are hyperbolas (centred at $(0,0)$ with asymptotes $y = \pm x$), and we have $1 \le x^2-y^2 \le 4$ when (x,y) lies in the region between the two hyperbolas: the set of such points (x,y) is the union of two connected regions; the region between the two right branches of the hyperbolas is given by $\sqrt{1+y^2} \le x \le \sqrt{4+y^2}$, and the region between the two left branches is given by $-\sqrt{4+y^2} \le x \le -\sqrt{1+y^2}$. The curve $xy = 0$ is the union of the two axes and the curve $xy = 1$ is a hyperbola (centred at $(0,0)$ with asymptotes along the axes), and the set of points (x,y) such that $0 \le xy \le 1$ is the union of two regions: the region in the first quadrant below $y = \frac{1}{x}$, and the region in the third quadrant above $y = \frac{1}{x}$. Thus the inverse image $f^{-1}(B)$ is the union of two connected regions: one in the right half-plane $x > 0$ given by $0 \le y \le \frac{1}{x}$ and $\sqrt{1+y^2} \le x \le \sqrt{4+y^2}$, and the other in the left half-plane $x < 0$ given by $\frac{1}{x} \le y \le 0$ and $-\sqrt{4+y^2} \le x \le \sqrt{1+y^2}$.

4: (a) Find an implicit equation, of the form $ax+by+cz=d$, for the tangent plane to the parametric surface $(x,y,z)=f(s,t)=\left(s-t^2\,,\,\frac{s}{t}\,,\,\sqrt{s\,t}\right)$ at the point where $(s,t)=(4,1)$.

Solution: We have $f(4,1)=\big(3,4,2\big)$ and

$$Df(s,t)=\begin{pmatrix} 1 & -2t \\ \frac{1}{t} & -\frac{s}{t^2} \\ \frac{\sqrt{t}}{2\sqrt{s}} & \frac{\sqrt{s}}{2\sqrt{t}} \end{pmatrix} \qquad Df(4,1)=\begin{pmatrix} 1 & -2 \\ 1 & -4 \\ \frac{1}{4} & 1 \end{pmatrix}$$

so the tangent plane is the plane through $p=(3,4,2)$ in the direction of the vectors $u=\left(1,1,\frac{1}{4}\right)$ and $v=(-2,-4,1)$. The plane has normal vector $w=u\times v=\left(2,-\frac{3}{2},-2\right)$, so the equation is of the form $2x-\frac{3}{2}y-2z=c$. We can put in $(x,y,z)=(3,4,2)$ to get $c=2\cdot 3-\frac{3}{2}\cdot 4-2\cdot 2=-4$, so the plane has equation $2x-\frac{3}{2}y-2z=-4$.

(b) Let C be the set of all $(u,v,w)\in\mathbb{R}^3$ such that the polynomial $f(x)=x^3+ux^2+vx+w$ has a triple real root, and let S be the set of all $(u,v,w)\in\mathbb{R}^3$ such that the polynomial. $f(x)=x^3+ux^2+vx+w$ has a multiple real root (that is a double or triple real root). Find a parametric equation for C and find a parametric equation and an implicit equation for S. As an optional additional exercise (not to be marked), use computer software to display the curve C and the surface S.

Solution: The monic polynomial with triple root t is $(x-t)^3=x^3-3t\,x^2+3t^2\,x-t^3$, so C is the twisted cubic curve given parametrically by

$$(u,v,w)=\alpha(t)=(-3t,3t^2,-t^3).$$

The monic polynomial with double root s and additional root t (possibly with $s=t$) is the polynomial $(x-s)^2(x-t)=x^3-(2s+t)x^2+(2st+s^2)x-s^2t$, so S is given parametrically by

$$(u,v,w)=\sigma(s,t)=(-(2s+t),s(s+2t),-s^2t).$$

We can eliminate the parameters s and t, for example as follows. From $u=-(2s-t)$ we obtain $t=-(u+2s)$. Then $v=s(s+2t)=s(s-2u-4s)=-s(2u+3s)$ and $w=-s^2t=s^2(u+2s)$, so we have $3w+2sv=s^2(3u+6s)-s^2(4u+6s)=-s^2u$, hence $9w+6sv-uv=-3s^2u+su(2u+3s)=2su^2$ so that $6sv-2su^2=uv-9w$. Thus when $3v\neq u^2$ we have $s=\frac{uv-9w}{2(3v-u^2)}$ and $t=-2s-u=-\frac{uv-9w}{3v-u^2}-u=\frac{9w-4uv+u^3}{3v-u^2}$. Note that $3v=u^2 \iff 3(2st+s^2)=(-2s-t)^2 \iff 6st+3s^2=4s^2+4st+t^2 \iff (s-t)^2=0 \iff s=t$. Conversely, when $s=\frac{uv-9w}{2(3v-u^2)}$ and $t=\frac{9w-4uv+u^3}{3v-u^2}$, a routine calculation shows that we have $u+(2s+t)=0$, $v-s(s+2t)=\frac{9(27w^2+4v^3+4u^3w-18uvw-u^2v^2)}{4(3v-u^2)^2}$ and $w+s^2t=\frac{(27-u^3)(27w^2+4v^3+4u^3w-18uvw-u^2v^2)}{4(3v-u^2)^3}$. So it appears that $S=\text{Null}(g)$ where

$$g(u,v,w)=27w^2+4v^3+4u^3w-18uvw-u^2v^2.$$

We outline how this can be verified: When $(u,v,w)=\sigma(s,t)$, a routine calculation shows that $g(u,v,w)=0$, and this shows that $\text{Range}(\sigma)\subseteq\text{Null}(g)$. When $g(u,v,w)=0$ with $3v\neq u^2$, the above calculations show that we can choose $s=\frac{uv-9w}{2(3v-u^2)}$ and $t=\frac{9w-4uv+u^3}{3v-u^2}$ to obtain $(u,v,w)=\sigma(s,t)$. Finally, when $g(u,v)=0$ with $3v=u^2$, we can choose $s=t=-\frac{1}{3}u$ to get $u=-3t$ and $v=\frac{1}{3}u^2=3t^2$ and $0=g(u,v,w)=27w^2+4(3t^2)^3-4(3t)^3(3t^2)+18(3t)(3t^2)w-(3t)^2(3t^2)^2=27(w^2+2t^3+t^6)=27(w+t^3)^2$ so that $w=-t^3$ giving $(u,v,w)=(-3t,3t^2,-t^3)=\sigma(t,t)=\sigma(s,t)$. This shows that $\text{Null}(g)\subseteq\text{Range}(\sigma)$.

We remark that S is a surface which has a cusp (or a fold) along the twisted cubic curve C.