MATH 247 Calculus 3, Solutions to Assignment 2

1: (a) Let $(u,v)=g(z)=\big(\sqrt{z-1},5\ln z\big)$, where $z=f(x,y)=4x^2-8xy+5y^2$. Use the Chain Rule to find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at the point $(2,1)$.

Solution: Write $h(x,y)=g(f(x,y))$. When $(x,y)=(2,1)$ we have $z=f(2,1)=5$ and so by the Chain Rule, we have $Dh(2,1)=Dg(5)\,Df(2,1)$, that is

$$\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{du}{dz}\\ \frac{dv}{dz}\end{pmatrix}\begin{pmatrix}\frac{\partial z}{\partial x}&\frac{\partial z}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{1}{2\sqrt{z-1}}\\ \frac{5}{z}\end{pmatrix}\begin{pmatrix}8x-8y&-8x+10y\end{pmatrix}=\begin{pmatrix}\frac14\\ 1\end{pmatrix}\begin{pmatrix}8&-6\end{pmatrix}=\begin{pmatrix}2&-\frac32\\ 8&-6\end{pmatrix}$$

(b) Let $(x,y)=f(r,\theta)=\big(r\cos\theta\,,\,r\sin\theta\big)$, let $z=g(x,y)$ and let $z=h(r,\theta)=g\big(f(r,\theta)\big)$. Suppose that $h(r,\theta)=r^2e^{\sqrt3(\theta-\frac{\pi}{6})}$. Find $Dg(\sqrt3,1)$.

Solution: Note that $(x,y)=(\sqrt3,1)$ when $(r,\theta)=\big(2,\frac{\pi}{6}\big)$ and then, by the Chain Rule,

$$\begin{aligned}Dh&=Dg\cdot Df\\ \begin{pmatrix}\frac{\partial h}{\partial r}&\frac{\partial h}{\partial\theta}\end{pmatrix}&=\begin{pmatrix}\frac{\partial g}{\partial x}&\frac{\partial g}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{pmatrix}\\ \Big(2r\,e^{\sqrt3(\theta-\frac{\pi}{6})},\ \sqrt3r^2e^{\sqrt3(\theta-\frac{\pi}{6})}\Big)&=\Big(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\Big)\begin{pmatrix}\cos\theta&-r\sin\theta\\ \sin\theta&r\cos\theta\end{pmatrix}\\ \big(4\,,\,4\sqrt3\big)&=\Big(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\Big)\begin{pmatrix}\frac{\sqrt3}{2}&-1\\ \frac12&\sqrt3\end{pmatrix}\end{aligned}$$

and so

$$\nabla g^T=\big(\tfrac{\partial g}{\partial x},\tfrac{\partial g}{\partial y}\big)=\big(4,3\sqrt3\big)\begin{pmatrix}\frac{\sqrt3}{2}&-1\\ \frac12&\sqrt3\end{pmatrix}^{-1}=\big(4,4\sqrt3\big)\cdot\frac12\begin{pmatrix}\sqrt3&1\\ -\frac12&\frac{\sqrt3}{2}\end{pmatrix}=\big(\sqrt3,5\big).$$

(c) Let $f(x,y,z)=x\,\sin(y^2-2xz)$ and let $\alpha(t)=\big(\sqrt t\,,\,\frac12t\,,\,e^{(t-4)/4}\big)$. Find the rate of change of f as we move along the curve $\alpha(t)$ when $t=4$.

Solution: We have $\alpha(4)=(2,2,1)$ and $f(\alpha(4))=f(2,2,1)=2\sin0=0$, and we have

$$\begin{aligned}Df(x,y,z)&=\big(\sin(y^2-2xz)-2xz\cos(y^2-2xz)\ ,\ 2xy\cos(y^2-2xz)\ ,\ -2x^2\cos(y^2-2xz)\big)\\ Df(2,2,1)&=(-4,8,-8)\\ \alpha'(t)&=\big(\tfrac{2}{2\sqrt t},\tfrac12,\tfrac14\,e^{(t-4)/4}\big)^T\\ \alpha'(4)&=\big(\tfrac14,\tfrac12,\tfrac14\big)^T.\end{aligned}$$

For $\beta(t)=f\big(\alpha(t)\big)$ we have $\beta'(t)=Df(\alpha(t))\alpha'(t)$, so the rate of change of f as we move along $\alpha(t)$ is

$$\beta'(4)=Df(2,2,1)\alpha'(4)=\big(-4,8,-8\big)\begin{pmatrix}\frac14\\ \frac12\\ \frac14\end{pmatrix}=1.$$

2: (a) Consider the surface $z = f(x,y)$ where $f(x,y) = \dfrac{4}{2 + x^4 + x^2 + y^2}$. An ant walks counterclockwise around the curve of intersection of the above surface $z = f(x,y)$ with the cylinder $(x-2)^2 + y^2 = 5$. Find the value of $\tan\theta$, where θ is the angle (from the horizontal) at which the ant is ascending when it is at the point $\left(1, 2, \frac{1}{2}\right)$.

Solution: We have $\frac{\partial f}{\partial x} = \frac{-16x^3 - 8x}{(2+x^4+x^2+y^2)^2}$ and $\frac{\partial f}{\partial y} = \frac{-8y}{(2+x^4+x^2+y^2)^2}$, and so we have $\frac{\partial f}{\partial x}(1,2) = -\frac{24}{64} = -\frac{3}{8}$ and $\frac{\partial f}{\partial y}(1,2) = -\frac{16}{64} = -\frac{1}{4}$. Thus

$$\nabla f(1,2) = \left(-\tfrac{3}{8}, -\tfrac{1}{4}\right)^T.$$

Looking down from above, the ant moves counterclockwise around the circle $(x-2)^2 + y^2 = 5$. The radius vector from the center $(2,0)$ to the point $(1,2)$ is the vector $r = (1,2) - (2,0) = (-1,2)$, and the unit tangent vector perpendicular to r, in the direction in which the ant moves, is $v = \frac{1}{\sqrt{5}}(-2,-1)$. If we parametrize the circle, in the counterclockwise direction, by $\alpha(t) = (x(t), y(t))$ with $\alpha(0) = (1,2)$, then we will have $\frac{\alpha'(0)}{|\alpha'(0)|} = v = \frac{1}{\sqrt{5}}(-2,-1)$. The ant moves along the curve $\gamma(t) = \big(x(t), y(t), z(t)\big) = \big(\alpha(t), f(\alpha(t))\big)$, the tangent vector at $t = 0$ is $\gamma'(0) = \big(\alpha'(0), Df(\alpha(0))\alpha'(0)\big)$. The angle of inclination θ of $\gamma'(0)$ is given by

$$\tan\theta = \frac{Df(\alpha(0))\alpha'(0)}{|\alpha'(0)|} = \nabla f(1,2) \bullet \frac{\alpha'(0)}{|\alpha'(0)|} = D_v f(1,2) = \left(-\tfrac{3}{8}, -\tfrac{1}{4}\right) \bullet \tfrac{1}{\sqrt{5}}\left(-2, -1\right) = \tfrac{1}{\sqrt{5}}.$$

(b) Consider the surface $z = f(x,y)$ where $f(x,y) = \dfrac{6x}{1 + x^2 + y^2}$. Show that any circle which passes through the points $(1,0)$ and $(-1,0)$ is a curve of steepest descent, that is for any point (x,y) on any circle C through the two points $(-1,0)$ and $(1,0)$, the slope of C at (x,y) is equal to the slope of the gradient vector $\nabla f(x,y)$.

Solution: The circle through $(1,0)$ and $(-1,0)$ with center at $(0,c)$ has equation $x^2 + (y-c)^2 = 1 + c^2$. Differentiate this equation (with respect to x) to get $2x + 2(y-c)y' = 0$. This shows that at the point (x,y) on the circle, the slope of the circle is $y' = -\frac{x}{y-c}$. On the other hand, $\nabla f(x,y) = \frac{6}{(1+x^2+y^2)^2}(1 - x^2 + y^2, -2xy)$, so the gradient vector has slope $m = -\frac{2xy}{1-x^2+y^2}$. When (x,y) is on the circle, we have $x^2 + (y-c)^2 = 1 + c^2$, so $x^2 + y^2 - 2cy = 1$, and so we can put $1 - x^2 = y^2 - 2cy$ into the formula for m to get $m = -\frac{2xy}{y^2 - 2cy + y^2} = \frac{-x}{y-c}$. Thus the slope of the circle at (x,y) is equal to the slope of the gradient vector at (x,y), as required.

3: (a) Find $\displaystyle\iint_D y\,e^x\,dA$ where D is the region in $\mathbb{R}^2$ bounded by $y=0$, $y=x$ and $x+y=2$.

Solution: We have

$$\begin{aligned}\iint_D y\,e^x\,dA &= \int_{y=0}^{1}\int_{x=y}^{2-y} y\,e^x\,dx\,dy = \int_{y=0}^{1}\Big[y\,e^x\Big]_{x=y}^{2-y}dy = \textstyle\int_{y=0}^{1} y\,e^{2-y} - ye^y\;dy\\ &= \Big[-(y+1)e^{2-y}-(y-1)e^y\Big]_{y=0}^{1} = e^2-2e-1.\end{aligned}$$

(b) Find $\displaystyle\iint_D \frac{x}{\sqrt{1+x^2+y^2}}\,dA$ where $D=\big\{(x,y)\,\big|\,0\le x\le 2\,,\;0\le y\le \tfrac12 x^2\big\}$.

Solution: Note that we can also write $D=\big\{(x,y)\,\big|\,0\le y\le 2\,,\;\sqrt{2y}\le x\le 2\big\}$ and so

$$\begin{aligned}\iint_D \frac{x}{\sqrt{1+x^2+y^2}}\,dA &= \int_{y=0}^{2}\int_{x=\sqrt{2y}}^{2}\frac{x}{\sqrt{1+x^2+y^2}}\,dx\,dy = \int_{y=0}^{2}\Big[\sqrt{1+x^2+y^2}\Big]_{x=\sqrt{2y}}^{2}dx\\ &= \int_{y=0}^{2}\sqrt{5+y^2}-\sqrt{1+2y+y^2}\;dy = \int_{y=0}^{2}\sqrt{5+y^2}-(1+y)\;dy\,.\end{aligned}$$

Using the substitution $\sqrt5\tan\theta=y$ so that $\sqrt5\sec\theta=\sqrt{5+y^2}$ and $\sqrt5\sec^2\theta\,d\theta=dy$, we have

$$\begin{aligned}\int\sqrt{5+y^2}\,dy &= \int 5\sec^3\theta\,d\theta = \tfrac52\big(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\big)+c\\ &= \tfrac52\Big(\tfrac{y\sqrt{5+y^2}}{5}+\ln\big(\tfrac{y+\sqrt{5+y^2}}{5}\big)\Big)+c = \tfrac12\,y\sqrt{5+y^2}+\tfrac52\ln\big(5+\sqrt{5+y^2}\big)+d\end{aligned}$$

(where $d=c-\frac52\ln5$). Thus

$$\begin{aligned}\iint_D\frac{x}{\sqrt{1+x^2+y^2}}\,dA &= \Big[\tfrac12\,y\sqrt{5+y^2}+\tfrac52\ln\big(5+\sqrt{5+y^2}\big)-y-\tfrac12 y^2\Big]_{y=0}^{2}\\ &= \big(\tfrac12\cdot6+\tfrac52\ln(2+3)-2-2\big)-\big(\tfrac52\ln\sqrt5\big)=\tfrac54\ln5-1.\end{aligned}$$

(c) Find $\displaystyle\iiint_D z\;dV$ where $D=\big\{(x,y,z)\,\big|\,0\le x\,,\;0\le y\le\sqrt{x^2+z^2}\,,\;0\le z\le\sqrt{1-x^2}\big\}$.

Solution: We have

$$\begin{aligned}\iiint_D z\;dV &= \int_{x=0}^{1}\int_{z=0}^{\sqrt{1-x^2}}\int_{y=0}^{\sqrt{x^2+z^2}} z\;dy\,dz\,dx = \int_{x=0}^{1}\int_{z=0}^{\sqrt{1-x^2}} z\sqrt{x^2+z^2}\;dz\,dx\\ &= \int_{x=0}^{1}\Big[\tfrac13(x^2+z^2)^{3/2}\Big]_{z=0}^{\sqrt{1-x^2}}dx = \int_{x=0}^{1}\tfrac13-\tfrac13x^3\;dx = \tfrac13-\tfrac1{12}=\tfrac14.\end{aligned}$$

4: (a) Find $\iint_D \cos(3x^2+y^2)\,dA$ where $D=\left\{(x,y)\,\middle|\,x^2+\frac{1}{3}\,y^2\le 1\right\}$.

Solution: The change of variables map $(x,y)=g(r,\theta)=\left(r\cos\theta\,,\ \sqrt{3}\,r\,\sin\theta\right)$ sends $C=[0,1]\times[0,2\pi]$ to the given region D, and we have $Dg=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sqrt{3}\sin\theta & \sqrt{3}r\cos\theta\end{pmatrix}$ so that $\det Dg=\sqrt{3}r$, and when $(x,y)=g(r,\theta)$ we have $3x^2+y^2=3r^2\cos^2\theta+3r^2\sin^2\theta=3r^2$, and so

$$\begin{aligned}\iint \cos(3x^2+y^2)\,dA&=\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\cos(3r^2)\sqrt{3}r\;d\theta\,dr=\int_{r=0}^{1}2\pi\sqrt{3}\,r\cos(3r^2)\,dr\\&=\left[\tfrac{\pi}{\sqrt{3}}\sin(3r^2)\right]_{r=0}^{1}=\tfrac{\pi}{\sqrt{3}}\sin 3.\end{aligned}$$

(b) Find $\iint_D e^{(y-x)/(y+x)}\,dA$ where D is the quadrilateral with vertices at $(1,1)$, $(2,0)$, $(4,0)$, $(2,2)$.

Solution: When $u=y+x$ and $v=y-x$ we have $y=\frac{u+v}{2}$ and $x=\frac{u-v}{2}$ and the lines $x+y=2$, $x+y=4$, $yt=0$ and $y=x$ (which form the boundary of D) are given by $u=2$, $v=4$, $u+v=0$ and $v=0$. So the change of variables map $(x,y)=g(u,v)=\left(\frac{u-v}{2},\frac{u+v}{2}\right)$ sends the set $C=\left\{(u,v)\,\middle|\,2\le u\le 4\,,\ -u\le v\le 0\right\}$ to the given region D. We have $Dg=\frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$ so that $\det Dg=\frac{1}{2}$, and so

$$\begin{aligned}\iint_D e^{(y-x)/(y+x)}\,dA&=\int_{u=2}^{4}\int_{v=-u}^{0}e^{v/u}\,dv\,du=\int_{u=2}^{4}\left[\tfrac{u}{2}\,e^{v/u}\right]_{v=-u}^{0}du\\&=\int_{u=2}^{4}\tfrac{u}{2}\left(1-\tfrac{1}{e}\right)du=\left[\tfrac{u^2}{4}\left(1-\tfrac{1}{e}\right)\right]_{u=2}^{4}=3\left(1-\tfrac{1}{e}\right).\end{aligned}$$

(c) Find $\iiint_D (x-y)z\,dV$ where $D=\left\{(x,y,z)\,\middle|\,x^2+y^2+z^2\le 4\,,\ z\ge\sqrt{x^2+y^2}\,,\ x\ge 0\right\}$.

Solution: The region D can be described in spherical coordinates by $0\le r\le 2$, $0\le\varphi\le\frac{\pi}{4}$, and $0\le\theta\le\pi$. In other words, the spherical coordinates map sends the set $C=\left\{(r,\varphi,\theta)\,\middle|\,0\le r\le 2\,,\ 0\le\varphi\le\frac{\pi}{4}\,,\ 0\le\theta\le\pi\right\}$ to the given region D. Thus we have

$$\begin{aligned}\iiint_D (x-y)z\,dV&=\int_{r=0}^{2}\int_{\varphi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{\pi}\big(r\sin\varphi\cos\theta-r\sin\varphi\sin\theta\big)\big(r\cos\varphi\big)\,r^2\sin\varphi\;d\theta\,d\varphi\,dr\\&=\int_{r=0}^{2}\int_{\varphi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{\pi}r^4\cdot\sin^2\varphi\cos\varphi\cdot(\cos\theta-\sin\theta)\;d\theta\,d\varphi\,dr\\&=\left(\int_{r=0}^{2}r^4\,dr\right)\left(\int_{\varphi=0}^{\frac{\pi}{4}}\sin^2\varphi\cos\varphi\;d\varphi\right)\left(\int_{\theta=0}^{\pi}(\cos\theta-\sin\theta)\;d\theta\right)\\&=\left[\tfrac{1}{5}\,r^5\right]_{r=0}^{2}\left[\tfrac{1}{3}\sin^3\varphi\right]_{\varphi=0}^{\frac{\pi}{4}}\left[\sin\theta+\cos\theta\right]_{\theta=0}^{\pi}=\tfrac{32}{5}\cdot\tfrac{1}{6\sqrt{2}}\cdot 2=\tfrac{16\sqrt{2}}{15}.\end{aligned}$$

5: (a) Find the total charge in the region $D=\left\{(x,y,z)\Big|\sqrt{\tfrac{1}{3}(x^2+y^2)}\le z\le\sqrt{4-x^2-y^2}\right\}$ where the charge density (charge per unit volume) is given by $f(x,y,z)=x^2$.

Solution: We use spherical coordinates $(x,y,z)=\big(r\sin\phi\cos\theta,r\sin\phi\sin\theta,r\cos\phi\big)$. Some students will see immediately that the cone $z=\sqrt{\frac{1}{3}(x^2+y^2)}$ is given in spherical coordinates by $\phi=\frac{\pi}{3}$. If you do not see this immediately, then you can verify this algebraically as follows. We have

$$x^2+y^2=(r\sin\phi\cos\theta)^2+(r\sin\phi\sin\theta)^2=r^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)=r^2\sin^2\phi$$

and so (since $r\ge0$ and $\sin\phi\ge0$)

$$z=\sqrt{\tfrac{1}{3}(x^2+y^2)}\iff r\cos\phi=\sqrt{\tfrac{1}{3}r^2\sin^2\phi}=\tfrac{1}{\sqrt{3}}r\sin\phi\iff\tan\phi=\sqrt{3}\iff\phi=\tfrac{\pi}{3}.$$

Thus the region D is described in spherical coordinates by $0\le r\le2$, $0\le\phi\le\frac{\pi}{3}$ and $0\le\theta\le2\pi$. Thus the total charge is

$$\begin{aligned}Q&=\iiint_D x^2\,dV=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\int_{\theta=0}^{2\pi}(r\sin\phi\cos\theta)^2\cdot r^2\sin\phi\,d\theta\,d\phi\,dr\\&=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\int_{\theta=0}^{2\pi}r^4\sin^3\phi\cos^2\theta\,d\theta\,d\phi\,dr=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\pi\,r^4\sin^3\phi\,d\phi\,dr\\&=\int_{r=0}^{2}\int_{\phi=0}^{\pi/3}\pi\,r^4(1-\cos^2\phi)\sin\phi\,d\phi\,dr=\int_{r=0}^{2}\pi\,r^4\Big[-\cos\phi+\tfrac{1}{3}\cos^3\phi\Big]_{\phi=0}^{\pi/3}dr\\&=\int_{r=0}^{2}\pi\,r^4\Big(\big(-\tfrac{1}{2}+\tfrac{1}{24}\big)-\big(-1+\tfrac{1}{3}\big)\Big)\,dr=\int_{r=0}^{2}\pi\,r^4\cdot\tfrac{-12+1+24-8}{24}\,dr\\&=\int_{r=0}^{2}\tfrac{5\pi}{24}\,r^4\,dr=\Big[\tfrac{\pi}{24}\,r^5\Big]_{r=0}^{2}=\tfrac{32\pi}{24}=\tfrac{4\pi}{3}\,.\end{aligned}$$

(b) Find the mass of the sphere $x^2+y^2+z^2=1$ when the density (mass per unit area) is given by $f(x,y,z)=3-z$.

Solution: The sphere is the image of the map $\sigma:[0,\pi]\times[0,2\pi]\to\mathbb{R}^3$ given by $(x,y,z)=\sigma(\varphi,\theta)=\big(\sin\varphi\cos\theta\,,\,\sin\varphi\sin\theta\,,\,\cos\varphi\big)$. We have

$$D\sigma=\big(\sigma_\varphi,\sigma_\theta\big)=\begin{pmatrix}\cos\varphi\cos\theta&-\sin\varphi\sin\theta\\\cos\varphi\sin\theta&\sin\varphi\cos\theta\\-\sin\varphi&0\end{pmatrix}\quad\text{and}\quad\sigma_\varphi\times\sigma_\theta=\begin{pmatrix}\sin^2\varphi\\\sin^2\varphi\sin\theta\\\sin\varphi\cos\varphi\end{pmatrix}$$

and hence $\big|\sigma_\varphi\times\sigma_\theta\big|=|\sin\varphi|\sqrt{\sin^2\varphi\cos^2\theta+\sin^2\varphi\sin^2\theta+\cos^2\varphi}=|\sin\varphi|=\sin\varphi$ (since $0\le\varphi\le\pi$). Thus the mass is given by

$$M=\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi}(3-\cos\varphi)\sin\varphi\,d\theta\,d\varphi=2\pi\int_{\varphi=0}^{\pi}3\sin\varphi-\sin\varphi\cos\varphi\,d\varphi=12\pi.$$

(c) Find the mass of the curve of intersection of the parabolic sheet $z=x^2$ with the paraboloid $z=2-x^2-2y^2$ when the density (mass per unit length) is given by $f(x,y,z)=|xy|$.

Solution: Let us find a parametric equation for the curve C of intersection. To get $z=x^2$ and $z=2-x^2-2y^2$, we need $x^2=2-x^2-2y^2$, that is $x^2+y^2=1$. Thus we can write $(x,y)=(\cos t,\sin t)$ with $t\in[0,2\pi]$. We also need $z=x^2$, so the curve C is given parametrically by $(x,y,z)=\alpha(t)=\big(\cos t,\sin t,\cos^2t\big)$. We have $\alpha'(t)=\big(-\sin t,\cos t,-2\sin t\cos t\big)$ and $\big|\alpha'(t)\big|=\sqrt{\sin^2t+\cos^2t+4\sin^2t\cos^2t}=\sqrt{1+\sin^2(2t)}$. Using the substitution $u=\cos(2t)$ so $du=-2\sin(2t)\,dt$, the mass is given by

$$\begin{aligned}M&=\int_{t=0}^{2\pi}\big|\cos t\sin t\big|\sqrt{1+\sin^2(2t)}\,dt=\int_{t=0}^{2\pi}\big|\tfrac{1}{2}\sin(2t)\big|\sqrt{1+\sin^2(2t)}\,dt=8\int_{t=0}^{\frac{\pi}{2}}\tfrac{1}{2}\sin(2t)\sqrt{1+\sin^2(2t)}\,dt\\&=\int_{t=0}^{\pi/2}4\sin(2t)\sqrt{2-\cos^2(2t)}\,dt=\int_{u=1}^{0}-2\sqrt{2-u^2}\,du=2\int_{u=0}^{1}\sqrt{2-u^2}=\tfrac{\pi}{2}+1\end{aligned}$$

(the final value was obtained by noticing that the integral $\int_0^1\sqrt{2-u^2}\,du$ measures the area of a region consisting of one eighth of the disc of radius $\sqrt{2}$ along with a triangle of base 1 and height 1).