MATH 247 Calculus 3, Assignment 3 Due Wed Oct 23, 11:00 pm

1: (a) Find $\displaystyle\iint_D \frac{dA}{\sqrt{x^2+y^2}}$ where $D=\big\{(x,y)\in\mathbb{R}^2\,\big|\,4\le x^2+y^2\le 4x\big\}$.

(b) Find $\displaystyle\iiint_D e^{x-y+z}\,dV$ where $D=\big\{(x,y,z)\in\mathbb{R}^3\,\big|\,2\le x-y+z\le 3\,,\,-1\le x+2y\le 1\,,\,0\le x-z\le 2\big\}$.

(c) Find $\displaystyle\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx$.

2: (a) For $r>0$, let $A(r)=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\le x\,,\,0\le y\,,\,x^2+y^2\le r^2\big\}$, $B(r)=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\le x\le r\,,\,0\le y\le r\big\}$ and $C(r)=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\le x\,,\,0\le y\,,\,x^2+y^2\le 2r^2\big\}$, and note that $A(r)\subseteq B(r)\subseteq C(r)$. You may assume, without proof, that if $f:\mathbb{R}^2\to\mathbb{R}$ is continuous with $f(x,y)\ge 0$ for all (x,y) then $\int_{A(r)} f\le\int_{B(r)} f\le\int_{C(r)} f$. Use this fact for the function $f(x,y)=e^{-(x^2+y^2)}$ to find the value of $\int_0^\infty e^{-x^2}\,dx=\lim_{r\to\infty}\int_0^r e^{-x^2}\,dx$.

(b) Let $a_0,a_1,a_2,a_3\in\mathbb{R}^3$, let $u_k=a_k-a_0$ for $k=1,2,3$, let $A=\big(u_1,u_2,u_3\big)\in M_3(\mathbb{R})$, suppose $\det A\ne 0$, and let T be the tetrahedron in $\mathbb{R}^3$ with vertices a_k. Find a formula, in terms of A, for the volume of T and a formula, in terms of a_0 and A, for the charge of T when the charge density is given by $\rho(x,y,z)=x$.

3: For part (a) of this problem, note when $g:\mathbb{R}^3\to\mathbb{R}^3$ is a change of variables map with Dg orthogonal, that is with $Dg^TDg=I$, if S is the surface $S=\text{Range}(\sigma)$ where $\sigma:D\subseteq\mathbb{R}^2\to\mathbb{R}^3$, and T is the surface $T=g(S)=\text{Range}(g\circ\sigma)$ then, using the Chain Rule and the fact that $Dg^TDg=I$, we have

$$\text{Area}\,(T)=\int_D\sqrt{\det\big(D(g\circ\sigma)^TD(g\circ\sigma)\big)}=\int_D\sqrt{\det\big(D\sigma^TDg^TDgD\sigma\big)}=\int_D\sqrt{\det\big(D\sigma^TD\sigma\big)}=\text{Area}\,(S).$$

In particular, note that translations and rotations preserve surface area.

(a) Find the area of the portion of a sphere S of radius R which lies between two parallel planes which intersect with the sphere and are separated by a distance d. Note that since translations and rotations preserve area, we can take S to be given by $x^2+y^2+z^2=R^2$ and we can take the planes to be given by $z=a$ and $z=b$ with $b-a=d$.

(b) A point p is chosen (uniformly) at random on the surface of the sphere S given by $(x+1)^2+y^2+z^2=1$ and a point q is chosen (uniformly and independently) at random on the surface of the sphere T given by $(x-1)^2+y^2+z^2=1$. Find the probability P that the distance between p and q is at most 1: if $\rho(x,y,z)$ is the probability that a point p chosen at random on S lies within 1 unit of $q=(x,y,z)$, then $P=\frac{1}{4\pi}\int_T\rho$.

4: (a) Let $A=\big\{(x,y)\in\mathbb{R}^2\big|\,y>x^2\big\}$. Prove, from the definition of an open set, that A is open in $\mathbb{R}^2$.

(b) Define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t)=\big(\sin t\,,\,t\,e^t\big)$. Prove, from the definition, that Range(f) is not closed in $\mathbb{R}^2$.

(c) Let A be the set of real numbers $x\in[0,1)$ which can be written in base 3 without using the digit 2, or in other words, let A be the set of real numbers of the form $x=\sum_{k=1}^\infty\frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$. Determine whether A is open or closed (or neither) in $\mathbb{R}$.