MATH 247 Calculus 3, Solutions to Assignment 3

1: (a) Find $\displaystyle\iint_D \frac{dA}{\sqrt{x^2+y^2}}$ where $D=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,4\le x^2+y^2\le 4x\right\}$.

Solution: Note that D is the region outside the circle $x^2+y^2=4$, and inside the circle $x^2+y^2=4x$, that is the circle $(x-2)^2+y^2=4$. These two circles intersect at $(x,y)=(1\pm\sqrt{3})$. The two circles are given in polar coordinates by $r=2$ and $r=4\cos\theta$ and these intersect at $(r,\theta)=\left(2,\pm\frac{\pi}{3}\right)$. Thus $D=g(C)$ where g is the polar coordinates map with $|\det Dg|=r$, and $C=\left\{(r,\theta)\in\mathbb{R}^2\,\middle|\,-\frac{\pi}{3}\le\theta\le\frac{\pi}{3}\,,\,2\le r\le 4\cos\theta\right\}$. We have

$$\begin{aligned}\iint_D \frac{dx\,dy}{\sqrt{x^2+y^2}} &= \iint_C \frac{r\,dr\,d\theta}{r} = \int_{\theta=-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{r=2}^{4\cos\theta} dr\,d\theta = \int_{\theta=-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos\theta-2\;d\theta\\ &= \Big[4\sin\theta-2\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \left(2\sqrt{3}-\tfrac{2\pi}{3}\right)-\left(-2\sqrt{3}+\tfrac{2\pi}{3}\right)=4\sqrt{3}-\tfrac{4\pi}{3}.\end{aligned}$$

(b) Find $\displaystyle\iiint_D e^{x-y+z}\;dV$ where $D=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,2\le x-y+z\le 3\,,\,-1\le x+2y\le 1\,,\,0\le x-z\le 2\right\}$.

Solution: Let $u=x-y+z$, $v=x+2y$ and $w=x-z$, that is

$$\begin{pmatrix}u\\v\\w\end{pmatrix}=A\begin{pmatrix}x\\y\\z\end{pmatrix}\qquad\text{where}\qquad A=\begin{pmatrix}1&-1&1\\1&2&0\\1&0&-1\end{pmatrix}.$$

Note that $g(C)=D$ where $C=[2,3]\times[-1,1]\times[0,2]=\left\{(u,v,w)\,\middle|\,2\le u\le 3\,,\,-1\le v\le 1\,,\,0\le w\le 2\right\}$ and $g:\mathbb{R}^3\to\mathbb{R}^3$ is the linear map given by $g=A^{-1}$. Since $\det A=-5$ we have $|\det Dg|=|\det A^{-1}|=\frac{1}{5}$ and so

$$\begin{aligned}\iiint_D e^{x-y+z}\;dx\,dy\,dz &= \iiint_C \tfrac{1}{5}\,e^u\;dw\,dv\,du = \int_{u=2}^{3}\int_{v=-1}^{1}\int_{w=0}^{2}\tfrac{1}{5}\,e^u\;dw\,dv\,du\\ &= \int_{u=2}^{3}\int_{v=-1}^{1}\tfrac{2}{5}\,e^u\,dv\,dw = \int_{u=2}^{3}\tfrac{4}{5}\,e^u\,du=\tfrac{4}{5}(e^3-e^2).\end{aligned}$$

(c) Find $\displaystyle\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx.$

Solution: Note that $g(C)=D$ where $D=\left\{(x,y,z)\,\middle|\,0\le x\le 1\,,\,0\le y\le\sqrt{1-x^2}\,,\,\sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2}\right\}$, $C=\left\{(r,\theta,z)\,\middle|\,0\le\theta\le\frac{\pi}{2}\,0\le r\le 1\,,\,r\le z\le\sqrt{2-r^2}\right\}$ and g is the cylindrical coordinates map with $|\det Dg|=r$. Thus

$$\begin{aligned}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx &= \iiint_D dx\,dy\,dz = \iiint_C r\;dr\,d\theta\,dz = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1}\int_{z=r}^{\sqrt{2-r^2}} r\;dz\,dr\,d\theta\\ &= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r\sqrt{2-r^2}-r^2\;dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}}\Big[-\tfrac{1}{3}(2-r^2)^{3/2}-\tfrac{1}{3}r^3\Big]_{r=0}^{1}d\theta\\ &= \int_{\theta=0}^{\frac{\pi}{2}} -\tfrac{1}{3}-\tfrac{1}{3}+\tfrac{1}{3}\cdot 2\sqrt{2}\;d\theta = \tfrac{\pi}{2}\left(-\tfrac{2}{3}+\tfrac{2\sqrt{2}}{3}\right)=\tfrac{\pi}{3}(\sqrt{2}-1).\end{aligned}$$

2: (a) For $r>0$, let $A(r)=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,0\le x\,,\,0\le y\,,\,x^2+y^2\le r^2\right\}$, $B(r)=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,0\le x\le r\,,\,0\le y\le r\right\}$ and $C(r)=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,0\le x\,,\,0\le y\,,\,x^2+y^2\le 2r^2\right\}$, and note that $A(r)\subseteq B(r)\subseteq C(r)$. You may assume, without proof, that if $f:\mathbb{R}^2\to\mathbb{R}$ is continuous with $f(x,y)\ge 0$ for all (x,y) then $\int_{A(r)}f\le\int_{B(r)}f\le\int_{C(r)}f$. Use this fact for the function $f(x,y)=e^{-(x^2+y^2)}$ to find the value of $\int_0^\infty e^{-x^2}\,dx=\lim\limits_{r\to\infty}\int_0^r e^{-x^2}\,dx$.

Solution: Let $I(r)=\int_{x=0}^r e^{-x^2}\,dx=\int_{y=0}^r e^{-y^2}\,dy$. Note that since $e^{-(x^2+y^2)}=e^{-x^2}e^{-y^2}$, we have

$$\int_{B(r)}f=\int_{x=0}^r\int_{y=0}^r e^{-x^2}e^{-y^2}\,dy\,dx=\int_{x=0}^r e^{-x^2}\int_{y=0}^r e^{-y^2}\,dy\,dx=\int_{x=0}^r e^{-x^2}dx\cdot\int_{y=0}^r e^{-y^2}dy=I(r)^2.$$

Using polar coordinates $(x,y)=g(s,\theta)=(s\cos\theta,s\sin\theta)$ with $\det Dg=s$, we have

$$\int_{A(r)}f=\int_{s=0}^r\int_{\theta=0}^{\frac{\pi}{2}}se^{-s^2}\,d\theta\,ds=\tfrac{\pi}{2}\int_{s=0}^r s\,e^{-s^2}\,ds=\tfrac{\pi}{2}\Big[-\tfrac12\,e^{-s^2}\Big]_{s=0}^r=\tfrac{\pi}{4}(1-e^{-r^2})\text{ and}$$

$$\int_{C(r)}f=\int_{s=0}^{\sqrt2\,r}\int_{\theta=0}^{\frac{\pi}{2}}se^{-s^2}\,d\theta\,ds=\tfrac{\pi}{2}\int_{s=0}^{\sqrt2\,r}s\,e^{-s^2}\,ds=\tfrac{\pi}{2}\Big[-\tfrac12\,e^{-s^2}\Big]_{s=0}^{\sqrt2\,r}=\tfrac{\pi}{4}(1-e^{-2r^2})$$

Since $\int_{A(r)}f\le\int_{B(r)}f\le\int_{C(r)}f$ we have $\frac{\pi}{4}(1-e^{-r^2})\le I(r)^2\le\frac{\pi}{4}(1-e^{-2r^2})$. By the Squeeze Theorem, we have $\lim\limits_{r\to\infty}I(r)^2=\frac{\pi}{4}$, and hence $\int_0^\infty e^{-x^2}dx=\lim\limits_{r\to\infty}I(r)=\frac{\sqrt{\pi}}{2}$.

(b) Let $a_0,a_1,a_2,a_3\in\mathbb{R}^3$, let $u_k=a_k-a_0$ for $k=1,2,3$, let $A=\big(u_1,u_2,u_3\big)\in M_3(\mathbb{R})$, suppose $\det A\ne 0$, and let T be the tetrahedron in $\mathbb{R}^3$ with vertices a_k. Find a formula, in terms of A, for the volume of T and a formula, in terms of a_0 and A, for the charge of T when the charge density is given by $\rho(x,y,z)=x$.

Solution: Let $g:\mathbb{R}^3\to\mathbb{R}^3$ be the affine map given by $g(t)=a_0+At$, that is by

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=g(r,s,t)=\begin{pmatrix}a_{0,1}\\a_{0,2}\\a_{0,3}\end{pmatrix}+A\begin{pmatrix}r\\s\\t\end{pmatrix}\quad\text{where}\quad A=\begin{pmatrix}u_{1,1}&u_{2,1}&u_{3,1}\\u_{1,2}&u_{2,2}&u_{3,2}\\u_{1,3}&u_{2,3}&u_{3,3}\end{pmatrix}$$

and note that $Dg=A$ and that $g(S)=T$ where S is the standard tetrahedron with vertices at $0,e_1,e_2,e_3$. We have

$$\begin{aligned}\mathrm{Vol}(S)&=\int_{r=0}^1\int_{s=0}^{1-r}\int_{t=0}^{1-r-s}dt\,ds\,dr=\int_{r=0}^1\int_{s=0}^{1-r}1-r-s\;ds\,dr\\&=\int_{r=0}^1\Big[(1-r)s-\tfrac12 s^2\Big]_{s=0}^{1-r}dr=\int_{r=0}^1\tfrac12(1-r)^2dr=\Big[-\tfrac16(1-r)^3\Big]_{r=0}^1=\tfrac16\\\mathrm{Vol}(T)&=\int_T 1=\int_S|\det A|=|\det A|\cdot\mathrm{Vol}(S)=\tfrac16\,|\det A|.\end{aligned}$$

Note that $\rho(g(r,s,t))$ is the first entry of $g(r,s,t)$, that is $\rho(g(r,s,t))=a_{0,1}+u_{1,1}r+u_{2,1}s+u_{3,1}t$, so the charge of T is

$$Q=\int_T\rho=\int_S(\rho\circ g)|\det Dg|=|\det A|\int_S(\rho\circ g)=|\det A|\int_S\big(a_{0,1}+u_{1,1}r+u_{2,1}s+u_{3,1}t\big)dr\,ds\,dt$$

We have

$$\begin{aligned}\int_S r&=\int_{r=0}^1\int_{s=0}^{1-r}\int_{t=0}^{1-r-s}r\;dt\,ds\,dr=\int_{r=0}^1\int_{s=0}^{1-r}r(1-r)-rs\;ds\,dr=\int_{r=0}^1\Big[r(1-r)s-\tfrac12 rs^2\Big]_{s=0}^{1-r}dr\\&=\int_{r=0}^1\tfrac12\,r(1-r)^2\,dr=\int_{r=0}^1\tfrac12(r-2r^2+r^3)\,dr=\tfrac12\Big[\tfrac12r^2-\tfrac23r^3+\tfrac14r^4\Big]_{r=0}^1\tfrac12\big(\tfrac12-\tfrac23+\tfrac14\big)=\tfrac1{24}\end{aligned}$$

and similarly (by interchanging the roles of r, s and t) $\int_S s=\int_S t=\frac1{24}$, and we have $\int_S 1=\mathrm{Vol}(S)=\frac16$, and hence

$$Q=|\det A|\Big(a_{0,1}\int_S 1+u_{1,1}\int_S r+u_{2,1}\int_S s+u_{3,1}\int_S t\Big)=|\det A|\big(\tfrac16 a_{0,1}+\tfrac1{24}(u_{1,1}+u_{2,1}+u_{3,1})\big).$$

This can also be written as

$$Q=\tfrac1{24}|\det A|\,\big(4a_0+A(e_1+e_2+e_3)\big)\cdot e_1.$$

3: (a) Find the area of the portion of a sphere S of radius R which lies between two parallel planes which intersect with the sphere and are separated by a distance d. Note that since translations and rotations preserve volume, we can take S to be given by $x^2+y^2+z^2=R^2$ and we can take the planes to be given by $z=a$ and $z=b$ with $b-a=d$.

Solution: We provide two solutions. For the first solutions, we describe the sphere parametrically by

$$(x,y,z)=\sigma(\theta,z)=\left(\sqrt{R^2-z^2}\,\cos\theta\,,\ \sqrt{R^2-z^2}\,\sin\theta\,,\ z\right)^T$$

with $\theta\in[0,2\pi]$ and $z\in[-1,1]$. Then we have

$$D\sigma=\left(\sigma_\theta,\sigma_z\right)=\begin{pmatrix}-\sqrt{R^2-z^2}\sin\theta & \frac{-z}{\sqrt{R^2-r^2}}\cos\theta\\ \sqrt{R^2-z^2}\,\cos\theta & \frac{-z}{\sqrt{R^2-r^2}}\,\sin\theta\\ 0 & 1\end{pmatrix}\quad\text{so that}\quad \sigma_\theta\times\sigma_z=\begin{pmatrix}\sqrt{R^2-z^2}\cos\theta\\ \sqrt{R^2-z^2}\sin\theta\\ z\end{pmatrix}$$

and hence $\left\|\sigma_\theta\times\sigma_z\right\|=\sqrt{(R_2-z^2)+z^2}=R$. Thus the area is

$$A=\int_{\theta=0}^{2\pi}\int_{z=a}^{b}\|\sigma_\theta\times\sigma_z\|\,dz\,d\theta=\int_{\theta=0}^{2\pi}\int_{z=a}^{b}R\ dz\,d\theta=\int_{\theta=0}^{2\pi}R(b-a)=2\pi R(b-a)=2\pi R\,d.$$

For the second solution, we describe the sphere parametrically by

$$(x,y,z)=\sigma(\varphi,\theta)=\left(R\sin\varphi\cos\theta\,,\ R\sin\varphi\sin\theta\,,\ R\cos\varphi\right)^T$$

with $\varphi\in[0,\pi]$ and $\theta\in[0,2\pi]$. Then we have

$$D\sigma=\left(\sigma_\varphi\,,\ \sigma_\theta\right)=\begin{pmatrix}R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta\\ R\cos\varphi\sin\theta & R\sin\varphi\cos\theta\\ -R\sin\varphi & 0\end{pmatrix}\quad\text{so that}\quad \sigma_\varphi\times\sigma_\theta=\begin{pmatrix}R^2\sin^2\varphi\cos\theta\\ R^2\sin^2\varphi\sin\theta\\ R^2\sin\varphi\cos\varphi\end{pmatrix}$$

hence

$$\left\|\sigma_\varphi\times\sigma_\theta\right\|=\sqrt{R^4\sin^4\varphi+R^4\sin^2\varphi\cos^2\varphi}=\sqrt{R^4\sin^2\varphi}=R^2\sin\varphi.$$

Note that we have $z=a$ when $R\cos\varphi=a$, that is when $\varphi=\cos^{-1}\frac{a}{R}$ and $z=b$ when $\varphi=\cos^{-1}\frac{b}{R}$, and so

$$A=\int_{\varphi=\cos^{-1}\frac{b}{R}}^{\cos^{-1}\frac{a}{R}}\int_{\theta=0}^{2\pi}R^2\sin\varphi\,d\theta\,d\varphi=\int_{\varphi=\cos^{-1}\frac{b}{R}}^{\cos^{-1}\frac{a}{R}}2\pi R^2\sin\varphi\,d\varphi=2\pi R^2\Big[-\cos\varphi\Big]_{\cos^{-1}\frac{b}{R}}^{\cos^{-1}\frac{a}{R}}=2\pi R^2\left(\tfrac{b}{R}-\tfrac{a}{R}\right)=2\pi R\,d.$$

(b) A point p is chosen (uniformly) at random on the surface of the sphere S given by $(x+1)^2+y^2+z^2=1$ and a point q is chosen (uniformly and independently) at random on the surface of the sphere T given by $(x-1)^2+y^2+z^2=1$. Find the probability P that the distance between p and q is at most 1: if $\rho(x,y,z)$ is the probability that a point p chosen at random on S lies within 1 unit of $q=(x,y,z)$, then $P=\frac{1}{4\pi}\int_T \rho$.

Solution: Let us call the portion of a sphere which lies between two parallel planes separated by a distance d a "slice of thickness d". When one of the two planes is tangent to the sphere, let us call the slice a "spherical cap". When $q=(x,y,z)$ lies on T so that we have $(x-1)^2+y^2=1$ (1), the distance r from q to the centre $(1,0,0)$ of S is given by $(x+1)^2+y^2=r^2$ (2). Subtract (1) from (2) to get $4x=r^2-1$ so that $r=\sqrt{1+4x}$. Note that the set of points p on S which lie within 1 unit of q is a spherical cap on S which is equal to a slice of thickness $d=1-\frac{r}{2}$ which, by part (a), has area $2\pi d=2\pi\big(1-\frac{r}{2}\big)$. The area of the entire sphere S (being a slice of thickness 2) is equal to 4π, so the probability that p lies within 1 unit of q is equal to

$$\rho(x,y,z)=\tfrac{1}{4\pi}\cdot 2\pi(1-\tfrac{r}{2})=\tfrac12-\tfrac14 r=\tfrac12-\tfrac14\sqrt{1+4x}$$

when $1\le r\le 2$, that is when $0\le x\le \frac34$, and is equal to 0 otherwise. When $(x,y,z)\in T$ we have $(x-1)^2+y^2+z^2=1$ so that $y^2+z^2=1-(x-1)^2=2x-x^2$, and so we can represent the sphere T parametrically by

$$(x,y,z)=\sigma(x,\theta)=\big(x\,,\ \sqrt{2x-x^2}\cos\theta\,,\ \sqrt{2x-x^2}\sin\theta\big)^T$$

with $x\in[0,2]$ and $\theta\in[0,2\pi]$. We have

$$D\sigma=\big(\sigma_x\,,\,\sigma_\theta\big)=\begin{pmatrix}1 & 0\\ \frac{1-x}{\sqrt{2x-x^2}}\cos\theta & -\sqrt{2x-x^2}\sin\theta\\ \frac{1-x}{\sqrt{2x-x^2}}\sin\theta & \sqrt{2x-x^2}\cos\theta\end{pmatrix}\quad\text{so that}\quad \sigma_x\times\sigma_\theta=\begin{pmatrix}1-x\\ -\sqrt{2x-x^2}\cos\theta\\ -\sqrt{2x-x^2}\sin\theta\end{pmatrix}$$

hence $\big\|\sigma_x\times\sigma_\theta\big\|=\sqrt{(1-x)^2+(2x-x^2)}=1$. The required probability is

$$\begin{aligned}P&=\tfrac{1}{4\pi}\int_T\rho=\tfrac{1}{4\pi}\int_{x=0}^{3/4}\int_{\theta=0}^{2\pi}\tfrac12-\tfrac14\sqrt{1+4x}\;dx=\int_{x=0}^{3/4}\tfrac14-\tfrac18\sqrt{1+4x}\;dx\\ &=\Big[\tfrac14 x-\tfrac{1}{48}(1+4x)^{3/2}\Big]_{x=0}^{3/4}=\big(\tfrac{3}{16}-\tfrac16\big)-\big(-\tfrac{1}{48}\big)=\tfrac{1}{24}\,.\end{aligned}$$

4: (a) Let $A = \big\{(x,y)\in\mathbb{R}^2 \big| y > x^2\big\}$. Prove, from the definition of an open set, that A is open in $\mathbb{R}^2$.

Solution: Let $(a,b)\in A$ so we have $b > a^2$ and hence $\sqrt{b} > |a|$. Let $r = \min\left(\frac{b-a^2}{2}\,,\,\frac{\sqrt{b}-|a|}{2}\right)$. We claim that $B\big((a,b),r\big)\subseteq A$. Let $(x,y)\in B\big((a,b),r\big)$. Note that

$$|x-a| \le \sqrt{(x-a)^2+(y-b)^2} = d\big((a,b),(x,y)\big) < r \le \tfrac{\sqrt{b}-|a|}{2}$$

and similarly

$$|y-b| < r \le \tfrac{b-a^2}{2}.$$

It follows that $|x|-|a| \le |x-a| < \frac{\sqrt{b}-|a|}{2}$ so that $|x| \le \frac{\sqrt{b}+|a|}{2}$ and that $b-y \le |y-b| < \frac{b-a^2}{2}$ so that $y > \frac{b+a^2}{2}$. Note that $0 \le \big(\sqrt{b}-|a|\big)^2 = b+a^2-2|a|\sqrt{b}$ so we have $2|a|\sqrt{b} \le b+a^2$. It follows that

$$x^2 < \left(\tfrac{\sqrt{b}+|a|}{2}\right)^2 = \tfrac{b+a^2+2|a|\sqrt{b}}{4} \le \tfrac{b+a^2}{2} < y.$$

Since $y > x^2$ we have $(x,y)\in A$. This shows that $B\big((a,b),r\big)\subseteq A$, as claimed, and so S is open.

(b) Define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t) = \big(\sin t\,,\,te^t\big)$. Prove, from the definition, that $\mathrm{Range}(f)$ is not closed in $\mathbb{R}^2$.

Solution: We need to show that $\mathrm{Range}(f)^c$ is not open. To say that $\mathrm{Range}(f)$ is open means that for all $a\in\mathrm{Range}(f)^c$ there exists $r>0$ such that $B(a,r)\subseteq\mathrm{Range}(f)^c$, or equivalently that for all $a\in\mathrm{Range}(f)^c$ there exists $r>0$ such that $B(a,r)\cap\mathrm{Range}(f)=\emptyset$. Thus we need to show that there exists $a\in\mathrm{Range}(f)^c$ such that for all $r>0$ we have $B(a,r)\cap\mathrm{Range}(f)\ne\emptyset$. Choose $a=(1,0)$. We claim that $a\in\mathrm{Range}(f)^c$. Suppose, for a contradiction, that $a\in\mathrm{Range}(f)$, say $a=f(t)$, that is $(1,0)=\big(\sin t\,,\,te^t\big)$. Since $\sin t = 1$, we have $t=\frac{\pi}{2}+2\pi n$ for some $n\in\mathbb{Z}$, so in particular we have $t\ne 0$. But since $te^t=0$ and $e^t\ne 0$ we must have $t=0$, and this gives the desired contradiction. Thus $a=(1,0)\in\mathrm{Range}(f)^c$, as claimed. It suffices to show that for all $r>0$ we have $B(a,r)\cap\mathrm{Range}(f)\ne\emptyset$. Let $r>0$. Note that, by l'Hôpital's Rule, we have

$$\lim_{t\to-\infty} t\,e^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \lim_{t\to-\infty} -e^t = 0,$$

so we can choose $R<0$ such that for all $t<R$ we have $|te^t|<r$. Choose $n\in\mathbb{Z}^+$ such that $\frac{\pi}{2}-2n\pi<R$ and let $t=\frac{\pi}{2}-2n\pi$. Then we have $\sin t=1$ and we have $|te^t|<r$ so that $\big\|f(t)-a\big\| = \big\|(1,te^t)-(1,0)\big\| = |te^t| < r$. Thus $f(t)\in B(a,r)$ and hence $B(a,r)\cap\mathrm{Range}(f)\ne\emptyset$, as required.

(c) Let A be the set of real numbers $x\in[0,1)$ which can be written in base 3 without using the digit 2, or in other words, let A be the set of real numbers of the form $x=\sum_{k=1}^\infty \frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$. Determine whether A is open or closed (or neither) in $\mathbb{R}$.

Solution: We claim that A is closed. Let A_n be the set of all $x\in[0,1)$ of the form $x=\sum_{k=1}^\infty\frac{a_k}{3^k}$ with $a_1,a_2,\cdots,a_n\in\{0,1\}$ and $a_k\in\{0,1,2\}$ for $k>n$. Note that $x\in A_n$ if and only if $a=b+t$ for some b of the form $b=\sum_{k=1}^n\frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$ and for some t of the form $t=\frac{1}{3^{n+1}}\sum_{k=0}^\infty\frac{a_k}{3^k}$ with each $a_k\in\{0,1,2\}$, or equivalently for some $t\in\big[0,\frac{1}{3^{n+1}}\big]$. Thus A_n is the union of the 2^n closed intervals of the form $\big[b,b+\frac{1}{3^{n+1}}\big]$, where $b=\sum\limits_{k=1}^n\frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$. For example, we have $A_1=\big[0,\frac13\big]\cup\big[\frac13,\frac23\big]=\big[0,\frac23\big]$ and $A_2=\big[0,\frac19\big]\cup\big[\frac19,\frac29\big]\cup\big[\frac13,\frac49\big]\cup\big[\frac49,\frac59\big]=\big[0,\frac29\big]\cup\big[\frac13,\frac59\big]$. Since $A=\bigcap_{n=1}^\infty A_n$ and each set A_n is closed, it follows that A is closed (by Theorem 2.14, which follows easily from Theorem 2.13), as claimed.

We remark that $A=\frac12 C=\big\{\frac12 x\big|x\in C\big\}$ where C is the famous Cantor set, which is the set of $x\in[0,1]$ which can be written in the form $x=\sum_{k=1}^\infty\frac{a_k}{3^k}$ with each $a_k\in\{0,2\}$. One can prove that C is closed in the same way that we proved that A is closed.