MATH 247 Calculus 3, Solutions to Assignment 4

1: (a) Let $A, B \subseteq \mathbb{R}^n$. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution: Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ we have $A \cup B \subseteq \overline{A} \cup \overline{B}$. Since $A \cup B \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B}$ is closed, it follows (from Definition 2.15) that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Note that for $X, Y \subseteq \mathbb{R}^n$, if $X \subseteq Y$ then every closed set containing Y also contains X, and so $\overline{X} \subseteq \overline{Y}$ (by Definition 2.15). Since $A \subseteq A \cup B$ we have $\overline{A} \subseteq \overline{A \cup B}$. Since $B \subseteq A \cup B$ we have $\overline{B} \subseteq \overline{A \cup B}$. Since $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ we have $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

(b) Let $A \subseteq \mathbb{R}^n$. Show that $A' = \overline{A}'$ or, in other words, show that A and $\overline{A}$ have the same limit points.

Solution: Note first that if $A \subseteq B$ then we have $A' \subseteq B'$: indeed if $a \in A'$ then given $r > 0$ we have $B^*(a,r) \cap B \supseteq B^*(a,r) \cap A \neq \emptyset$. Since $A \subseteq \overline{A}$, it follows that $A' \subseteq \overline{A}'$. It remains to show that $\overline{A}' \subseteq A'$. Let $a \in \overline{A}'$. Let $r > 0$. We must show that $B^*(a,r) \cap A \neq \emptyset$. Since $a \in \overline{A}'$ we can choose an element $x \in B^*\big(a, \frac{r}{2}\big) \cap \overline{A}$. Since $x \in \overline{A} = A \cup A'$, either we have $x \in A$ or we have $x \in A'$. If $x \in A$ then we have $x \in B^*(a,r) \cap A$ so that $B^*(a,r) \cap A \neq \emptyset$. Suppose that $x \in A'$. Let $s = d(x,a)$ and note that since $x \in B^*\big(a, \frac{r}{2}\big)$ we have $0 < s < \frac{r}{2}$. Since $x \in A'$ we can choose $y \in B^*(x,s) \cap A$. Then we have $y \in A$, and we have $y \neq a$ $\big($since $d(y,x) < s = d(a,x)\big)$, and we have $d(y,a) \leq d(y,x) + d(x,a) < s + \frac{r}{2} < r$, and hence $y \in B^*(a,r) \cap A$ so that $B^*(a,r) \cap A \neq \emptyset$, as required.

(c) Let $A, B \subseteq \mathbb{R}^n$ be disjoint closed sets. Show that there exist disjoint open sets $U, V \subseteq \mathbb{R}^n$ with $A \subseteq U$ and $B \subseteq V$.

Solution: Let A and B be disjoint closed sets in $\mathbb{R}^n$. For each $a \in A$, since $A \cap B = \emptyset$ we have $a \in B^c$, and since B is closed so that B^c is open, we can choose $r_a > 0$ such that $B(a, 2r_a) \subseteq B^c$, that is $B(a, 2r_a) \cap B = \emptyset$. Similarly, for each $b \in B$ we can choose $s_b > 0$ such that $B(b, 2s_b) \subseteq A^c$, that is $B(b, 2s_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} B(a, r_a)$ and $V = \bigcup_{b \in B} B(b, s_b)$. Then U and V are open with $A \subseteq U$ and $B \subseteq V$. We claim that $U \cap V = \emptyset$. Suppose, for a contradiction, that $c \in U \cap V$. Since $c \in U = \bigcup_{a \in A} B(a, r_a)$ we can choose $a \in A$ such that $c \in B(a, r_a)$. Since $c \in V = \bigcup_{b \in B} B(b, s_b)$ we can choose $b \in B$ such that $c \in B(b, s_b)$. If $r_a \leq s_b$ then $d(a,b) \leq d(a,c) + d(c,b) < r_a + s_b \leq 2s_b$ so that $a \in B(b, 2s_b)$, but this contradicts the fact that $B(b, 2s_b) \cap A = \emptyset$. Similarly, if $s_b \leq r_a$ then $d(a,b) < 2r_a$ so that $b \in B(a, 2r_a)$, contradicting the fact that $B(a, 2r_a) \cap B = \emptyset$. Thus $U \cap V = \emptyset$, as claimed.

2: (a) Let $A, B \subseteq \mathbb{R}^n$. Show that $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

Solution: Suppose that $x \notin (\partial A \cup \partial B)$, that is $x \notin \partial A = \overline{A} \setminus A^o$ and $x \notin \partial B = \overline{B} \setminus B^o$. This means that $\big(x \notin \overline{A}$ or $x \in A^o\big)$ and $\big(x \notin \overline{B}$ or $x \in B^o\big)$. If $x \in A^o$ then we can choose $r > 0$ so that $B(a,r) \subseteq A$, and then we also have $B(a,r) \subseteq (A \cup B)$ so that $x \in (A \cup B)^o$, and hence $x \notin \partial(A \cup B)$. Similarly, if $x \in B^o$ then we also have $x \in (A \cup B)^o$ hence $x \notin \partial(A \cup B)$. Finally, if $x \notin \overline{A}$ and $x \notin \overline{B}$, then we have $x \notin \overline{A} \cup \overline{B}$ and hence, by Question 1(a), we have $x \notin \overline{A \cup B}$, so that again $x \notin \partial(A \cup B)$.

(b) Let $A, B \subseteq \mathbb{R}^n$. Show that $\partial(A \cap B) \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$. Hint: first show that if $a \in \partial(A \cap B)$ and $a \notin (A \cap \partial B)$ then $a \in \partial A$.

Solution: We claim that if $a \in \partial(A \cap B)$ and $a \notin (A \cap \partial B)$ then we have $a \in \partial A$. Suppose that $a \in \partial(A \cap B)$ and $a \notin (A \cap \partial B)$. We need to show that for all $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$. Let $r > 0$. Note that since $a \in \partial(A \cap B)$ we have $B(a,r) \cap (A \cap B) \neq \emptyset$. Since $A \cap B \subseteq A$ we have $B(a,r) \cap (A \cap B) \subseteq B(a,r) \cap A$. Since $B(a,r) \cap (A \cap B) \neq \emptyset$ and $B(a,r) \cap (A \cap B) \subseteq B(a,r) \cap A$, we also have $B(a,r) \cap A \neq \emptyset$. It remains to show that $B(a,r) \cap A^c \neq \emptyset$.

Since $a \notin (A \cap \partial B)$, either $a \notin A$ or $a \notin \partial B$. In the case that $a \notin A$ we have $a \in B(a,r) \cap A^c$ so that $B(a,r) \cap A^c \neq \emptyset$. Suppose that $a \notin \partial B$. Since $a \notin \partial B$ we can choose $s > 0$ such that either $B(a,s) \cap B = \emptyset$ or $B(a,s) \cap B^c = \emptyset$. Since $a \in \partial(A \cap B)$ we have $B(a,s) \cap (A \cap B) \neq \emptyset$, and hence (because $A \cap B \subseteq B$) we also have $B(a,s) \cap B \neq \emptyset$, and it follows that $B(a,s) \cap B^c = \emptyset$. Let $\delta = \min(r,s)$ and note that since $B(a,s) \cap B^c = \emptyset$ we also have $B(a,\delta) \cap B^c = \emptyset$. Since $a \in \partial(A \cap B)$ we have $B(a,\delta) \cap (A \cap B)^c \neq \emptyset$, that is $B(a,r) \cap (A^c \cup B^c) \neq \emptyset$, or equivalently $\big(B(a,\delta) \cap A^c\big) \cup \big(B(a,\delta) \cap B^c\big) \neq \emptyset$. Since we know that $B(a,\delta) \cap B^c = \emptyset$, it follows that $B(a,\delta) \cap A^c \neq \emptyset$, hence also $B(a,r) \cap A^c \neq \emptyset$, as required.

(c) Give an example of sets $A, B \subseteq \mathbb{R}$ for which $\partial(A \cap B) \neq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

Solution: One such example is obtained by letting $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Since $A \cap B = \emptyset$, we also have $\partial(A \cap B) = \emptyset$. Since A is dense in $\mathbb{R}$, it follows that $A' = \mathbb{R}$ so that $\overline{A} = \mathbb{R}$. Since $B = A^c$ is dense in $\mathbb{R}$, it follows that $A^o = \emptyset$ so that $\partial A = \overline{A} \setminus A^o = \mathbb{R}$. Similarly we have $\overline{B} = \mathbb{R}$ and $B^o = \emptyset$ and $\partial B = \mathbb{R}$. Thus $(A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B) = (\mathbb{R} \cap \mathbb{R}) \cup (\mathbb{R} \cap \mathbb{R}) \cup (\emptyset \cap \emptyset) = \mathbb{R}$.

3: (a) Let $a \in \mathbb{R}^n$, let $r > 0$, and let $B(a,r) \subseteq A \subseteq \overline{B}(a,r)$. Show that $A^o = B(a,r)$ and $\overline{A} = \overline{B}(a,r)$.

Solution: Since $B(a,r)$ is open and $B(a,r) \subseteq A$, it follows that $B(a,r) \subseteq A^o$. Let $b \in A^o$. We must show that $b \in B(a,r)$. Suppose, for a contradiction, that $b \notin B(a,r)$, that is $|b-a| \geq r$. Since $b \in A^o \subseteq A \subseteq \overline{B}(a,r)$, we have $|b-a| \leq r$, and hence $|b-a| = r$. Since $b \in A^o$ we can choose $s > 0$ so that $B(b,s) \subseteq A$. Let $x = b + \frac{s}{2r}(b-a)$ and note that $|x-b| = \frac{s}{2}$ so that $x \in B(b,s)$. Also note that $x - a = (b-a) + \frac{s}{2r}(b-a) = \big(1 + \frac{s}{2r}\big)(b-a)$, so that $|x-a| = \big(1+\frac{s}{2r}\big) \cdot r = r + \frac{s}{2} > r$, and hence $x \notin \overline{B}(a,r)$ so that $x \notin A$. But since $x \in B(b,s)$ with $x \notin A$, this contradicts the fact that $B(b,s) \subseteq A$. Thus $b \in B(a,r)$ as required.

Since $\overline{B}(a,r)$ is closed and $A \subseteq \overline{B}(a,r)$, it follows that $\overline{A} \subseteq \overline{B}(a,r)$. Let $b \in \overline{B}(a,r)$. We must show that $b \in \overline{A}$, that is $b \in A$ or $b \in A'$. Since $b \in \overline{B}(a,r)$ we have $|b-a| \leq r$. If $|b-a| < r$ then we have $b \in B(a,r) \subseteq A$ so that $b \in A$. Suppose that $|b-a| = r$. We claim that $b \in A'$. Let $s > 0$. Choose t with $0 < t < \max\{r,s\}$ and let $x = b - \frac{t}{r}(b-a)$. Then we have $|x-b| = \frac{t}{r}|b-a| = t$ with $0 < t < s$ so that $x \in B^*(b,s)$, and we have $|x-a| = \big|\big(1-\frac{t}{r}\big)(b-a)\big| = r - t < r$ so that $x \in B(a,r) \subseteq A$ Thus $x \in B^*(b,s) \cap A$ so that $B^*(b,s) \cap A \neq \emptyset$, which shows that $b \in A'$, as claimed.

(b) Determine whether for every subset $P \subseteq \mathbb{R}^n$, we have $\overline{B_P(a,r)} = \overline{B}_P(a,R)$ for all $a \in P$ and all $r > 0$.

Solution: This is false. For example, when $a, b \in \mathbb{R}^n$ with $a \neq b$, and $P = \{a,b\}$, and $r = |b-a|$, we have $B_P(a,r) = \{a\}$ which is closed (both in P and in $\mathbb{R}^n$) so that $\overline{B_P(a,r)} = \{a\}$, but we have $\overline{B}_P(a,r) = \{a,b\}$.

(c) Let $A \subseteq P \subseteq \mathbb{R}^n$. Prove that A is compact in P if and only if A is compact in $\mathbb{R}^n$.

Solution: Suppose that A is compact in P. Let T be an open cover for A in $\mathbb{R}^n$. For each $V \in T$, let $U_V = V \cap P$. By Theorem 2.31, each set U_V is open in P. Since $A \subseteq P$ and $A \subseteq \bigcup_{V \in T} V$, we also have $A \subseteq \bigcup_{V \in T}(V \cap P) = \bigcup_{V \in T} U_V$. Thus the set $S = \big\{U_V \,\big|\, V \in T\big\}$ is an open cover for A in P. Since A is compact in P we can choose a finite subcover, say $\big\{U_{V_1}, \cdots U_{V_n}\big\}$ of S, where each $V_i \in T$. Since $A \subseteq \bigcup_{i=1}^n U_{V_i} = \bigcup_{i=1}^n (V_i \cap P)$, we also have $A \subseteq \bigcup_{i=1}^n V_i$ and so $\{V_1, \cdots, V_n\}$ is a finite subcover of T.

Suppose, conversely, that A is compact in $\mathbb{R}^n$. Let S be an open cover for A in P. For each $U \in S$, by Theorem 2.31, we can choose an open set V_U in $\mathbb{R}^n$ such that $U = V_U \cap P$. Then $T = \big\{V_U \,\big|\, U \in S\big\}$ is an open cover of A in $\mathbb{R}^n$. Since A is compact in $\mathbb{R}^n$ we can choose a finite subcover, say $\big\{V_{U_1}, \cdots, V_{U_n}\big\}$ of T, where each $U_i \in S$. Then $A \subseteq \bigcup_{i=1}^n (V_{U_i} \cap P) = \bigcup_{i=1}^n U_i$ and so $\{U_1, \cdots, U_n\}$ is a finite subcover of S in P.

4: (a) Let $A \subseteq \mathbb{R}^n$ be compact and let S be an open cover of A. Show that there exists $r > 0$ such that for every $a \in A$ there exists $U \in S$ such that $B(a, r) \subseteq U$.

Solution: For each $p \in A$, since S is an open cover for A we can choose $U_p \in S$ with $p \in U_p$ and then, since U_p is open we can choose $r_p > 0$ so that $B(p, 2r_p) \subseteq U_p$. Note that the set $T = \big\{ B(p, r_p) \big| p \in A \big\}$ is an open cover for A. Since A is compact, we can choose a finite subcover, say $\big\{ B(p_1, r_{p_1}), \cdots, B(p_\ell, r_{p_\ell}) \big\}$ of T for A, with each $p_k \in A$. Let $r = \min\{r_{p_1}, \cdots, r_{p_\ell}\}$. We claim that for every $a \in A$ there exists $U \in S$ such that $B(a, r) \subseteq U$. Let $a \in A$. Choose an index k such that $a \in B(p_k, r_{p_k})$, and let $U = U_{p_k} \in S$. For all $x \in B(a, r)$ we have $|x - p_k| \leq |x - a| + |a - p_k| \leq r + r_{p_k} \leq 2r_{p_k}$ and hence $x \in B(p_k, 2r_{p_k}) \subseteq U_{p_k} = U$. This shows that $B(a, r) \subseteq U$, as required.

(b) Let $C_1, C_2, C_3, \cdots$ be non-empty closed sets in $\mathbb{R}^n$ with $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$. Show that if each set C_k is compact then $\bigcap\limits_{k=1}^{\infty} C_k \neq \emptyset$, and find an example where the sets C_k are not compact and we have $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$.

Solution: Suppose that each set C_k is compact, and suppose, for a contradiction, that $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$. Then

$$\mathbb{R}^n = \emptyset^{\,c} = \Big(\bigcap_{k=1}^{\infty} C_k \Big)^c = \bigcup_{k=1}^{\infty} {C_k}^c = {C_1}^c \cup \bigcup_{k=2}^{\infty} {C_k}^c .$$

It follows that $C_1 \subseteq \bigcup\limits_{k=2}^{\infty} {C_k}^c$ since given $a \in C_1$ we have $a \in {C_1}^c \cup \bigcup\limits_{k=2}^{\infty} {C_k}^c$ but $a \notin {C_1}^c$, and so $a \in \bigcup\limits_{k=2}^{\infty} {C_k}^c$. Thus $S = \big\{ {C_2}^c, {C_3}^c, {C_4}^c, \cdots \big\}$ is an open cover for C_1. Since C_1 is compact, we can choose a finite sub-cover $T = \big\{ {C_{k_1}}^c, {C_{k_2}}^c, \cdots, {C_{k_\ell}}^c \big\}$ say with $2 \leq k_1 < k_2 < \cdots < k_\ell$. Since T covers C_1 we have $C_1 \subseteq \bigcup\limits_{i=1}^{\ell} {C_{k_i}}^c$. Since $C_{k_1} \supseteq C_{k_2} \supset \cdots \supset C_{k_\ell}$ we have ${C_{k_1}}^c \subseteq {C_{k_2}}^c \subseteq \cdots \subseteq {C_{k_\ell}}^c$ and hence $\bigcup\limits_{i=1}^{\ell} {C_{k_i}}^c = {C_{k_\ell}}^c$. Thus we obtain $C_1 \subseteq {C_{k_\ell}}^c$, or equivalently $C_1 \cap C_{k_\ell} = \emptyset$. But this is not possible since $C_1 \cap C_{k_\ell} = C_{k_\ell} \neq \emptyset$.

Note that the sets $C_k = \mathbb{R}^m \setminus B(0, n)$ are closed in $\mathbb{R}^m$ with $C_1 \supseteq C_2 \supseteq \cdots$, but $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$.