MATH 247 Calculus 3, Assignment 5 — Due Mon Nov 18, 11:00 pm

1: Note that $\mathbb{C}=\mathbb{R}^2$ so a sequence in $\mathbb{C}$ is a sequence in $\mathbb{R}^2$.

(a) For $k\geq 0$, let $x_k=\left(\frac{3+i\sqrt{3}}{4}\right)^k\in\mathbb{C}$, and for $n\geq 0$, let $s_n=\sum\limits_{k=0}^{n}x_k\in\mathbb{C}$. Use the definition of the limit (for a sequence in $\mathbb{R}^2$) to find $a,b\in\mathbb{R}$ such that $\lim\limits_{n\to\infty}s_n=a+ib$.

(b) Let $c=\frac{2-i}{8}\in\mathbb{C}$. Let $(z_n)_{n\geq 0}$ be the sequence in $\mathbb{C}$ given by $z_0=0$ and $z_{n+1}={z_n}^2+c$ for $n\geq 0$. Determine whether $(z_n)_{n\geq 0}$ converges in $\mathbb{C}$ and, if so, find $\lim\limits_{n\to\infty}z_n$ in $\mathbb{C}$.

2: (a) Let $f(x,y)=\dfrac{xy^2}{x^2+2y^2}$ for $(x,y)\neq(0,0)$. Determine whether $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ exists and, if so, find it.

(b) Let $f(x,y)=\dfrac{x\sqrt{y}}{x^2+y}$ for $y>0$. Determine whether $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ exists and, if so, find it.

(c) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=\left\{\begin{array}{cl}\dfrac{xy}{x^2-y^2} & \text{if } y\neq\pm x\\ 0 & \text{if } y=\pm x\end{array}\right\}$. Determine where $f(x,y)$ is continuous, that is find all points $(a,b)\in\mathbb{R}^2$ such that f is continuous at (a,b).

3: For each of the following subsets $A\subseteq\mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A=\left\{(a,b,c,d)\in\mathbb{R}^4\,\middle|\,\begin{pmatrix}a&b\\c&d\end{pmatrix}^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$.

(b) A is the set of points $(a,b,c)\in\mathbb{R}^3$ such that the polynomial $p(x)=x^3+ax^2+bx+c$ has three distinct real roots which all lie in the closed interval $[-1,1]$.

4: (a) When $A\subseteq\mathbb{R}^\ell$ is unbounded, $f:A\subseteq\mathbb{R}^\ell\to\mathbb{R}^m$, and $b\in\mathbb{R}^m$, we write $\lim\limits_{x\to\infty}f(x)=b$ when

$$\forall\epsilon>0\ \exists r>0\ \ \forall x\in A\ \ \big(|x|\geq r\Longrightarrow|f(x)-b|<\epsilon\big).$$

Show that if $A\subseteq\mathbb{R}^\ell$ is closed and unbounded, and $f:A\subseteq\mathbb{R}^\ell\to\mathbb{R}^m$ is continuous, and $\lim\limits_{x\to\infty}f(x)=b\in\mathbb{R}^m$, then f is uniformly continuous on A.

(b) Show that if $f:A\subseteq\mathbb{R}^\ell\to\mathbb{R}^m$ is uniformly continuous on A then there exists a unique continuous function $g:\overline{A}\subseteq\mathbb{R}^\ell\to\mathbb{R}^m$ with $g(x)=f(x)$ for all $x\in A$, and that g is uniformly continuous on $\overline{A}$.