MATH 247 Calculus 3, Assignment 6 Due Mon Dec 2, 11:00 pm

1: (a) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(0,0)=0$ and $f(x,y)=\frac{x^3-xy^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Determine whether f is differentiable at $(0,0)$.

(b) Suppose $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ is differentiable and f has a local maximum at $a\in U$. Show that $Df(a)=O$ (this is Exercise 6.15 in the lecture notes).

(c) Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_k}{\partial x_\ell}(x)$ exist and are bounded in U. Prove that f is continuous.

2: (a) Let $(u,v)=f(x,y)=\Big(x\ln(y-x^4),\big(2+\frac{y}{x}\big)^{3/2}\Big)$. Explain why f is locally invertible in a neighbourhood of $(1,2)$ and find the linearization of its inverse at $(0,8)$.

(b) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=2x^3-3x^2+2y^3+3y^2$ and let $C=\text{Null}(f)$. Use the Implicit Function Theorem to find all the points on C at which C is locally equal to the graph of a function $y=g(x)$, or locally equal to the graph of a function $x=h(y)$.

3: Read Chapter 6 in the Lecture Notes, then solve the following problems.

(a) Let $U=\big\{(x,y)\in\mathbb{R}^2\big|\,x^2>y^2\big\}$. Find the 2^{nd} Taylor polynomial of the map $f:U\subseteq\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=\sqrt{x^2-y^2}$ at the point $(5,4)$.

(b) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=x^2-xy+y^3-y$. Find and classify all the critical points of f in $\mathbb{R}^2$, and find the absolute maximum and minimum values of f on the set $A=\big\{(x,y)\in\mathbb{R}^2\big|\,y^2-1\leq x\leq 2\big\}$.

4: (Lagrange Multipliers) For $X\subseteq\mathbb{R}^n$ with $a\in X$, we define the **tangent space** of X at a to be the set T_aX of all vectors $u\in\mathbb{R}^n$ such that there exists $\delta>0$ and there exists a differentiable map $\alpha:(-\delta,\delta)\subseteq\mathbb{R}\to X\subseteq\mathbb{R}^n$ with $\alpha(0)=a$ such that $\alpha'(0)=u$.

(a) Let $U\subseteq\mathbb{R}^n$ be open with $a\in U$, let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$ be differentiable at a, let $b=f(a)$. Prove that $T_{(a,b)}\text{Graph}(f)=\text{Graph}(Df(a))$, which is an n-dimensional vector space in $\mathbb{R}^{n+m}$.

(b) Let $U\subseteq\mathbb{R}^n$ be open with $a\in U$, let $g:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$ be $\mathcal{C}^1$ in U with $g(a)=0$, and suppose that $\text{rank}\big(Dg(a)\big)=m<n$. Prove that $T_a\text{Null}(g)=\text{Null}\big(Dg(a)\big)$, an $(n-m)$-dimensional vector space in $\mathbb{R}^n$.

(c) Let $U\subseteq\mathbb{R}^n$ be open with $a\in U$, let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ be differentiable at a, let $g:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$ be $\mathcal{C}^1$ in U with $g(a)=0$ and $\text{rank}\big(Dg(a)\big)=m<n$. Prove that if $f(a)\geq f(x)$ for all $x\in\text{Null}(g)$, or if $f(a)\leq f(x)$ for all $x\in\text{Null}(g)$, then $\nabla f(a)\in\text{Row}\big(Dg(a)\big)=\text{Span}\big\{\nabla g_1(a),\cdots,\nabla g_m(a)\big\}$.

(d) Using Part (c), find the maximum and minimum values of $f(x,y,z)=xy$ on the circle in $\mathbb{R}^3$ given by $x^2+y^2+z^2=1$ and $x+y+z=0$ $\big($first let $g(x,y)=\big(x^2+y^2+z^2-1\,,\,x+y+z\big)$ and find all points $a\in\mathbb{R}^3$ with $g(a)=0$ such that $\nabla f(a)\in\text{Row}(Dg(a))\big)$.