MATH 247 Calculus 3, Solutions to Assignment 6

1: (a) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(0,0)=0$ and $f(x,y)=\frac{x^3-xy^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Determine whether f is differentiable at $(0,0)$.

Solution: We claim that f is not differentiable at $(0,0)$. When $\alpha(t)=(t,0)$ and $g(t)=f(\alpha(t))=t$, we have $\frac{\partial f}{\partial x}(0,0)=g'(0)=1$. When $\beta(t)=(0,t)$ and $h(t)=f(\beta(t))=0$, we have $\frac{\partial f}{\partial y}(0,0)=h'(0)=0$. When $\gamma(t)=(t,t)$ and $k(t)=f(\gamma(t))=0$, if f was differentiable at $(0,0)$, then by the Chain Rule we would have $k'(0)=Df(0,0)\gamma'(0)=(1\ 0)\binom{1}{1}=1$, but instead we have $k'(0)=0$.

(b) Suppose $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ is differentiable and f has a local maximum at $a\in U$. Show that $Df(a)=O$ (this is Exercise 6.15 in the lecture notes).

Solution: Suppose, for a contradiction, that $Df(a)\neq O$. Choose $0\neq u\in\mathbb{R}^n$ such that $Df(a)u\neq 0$. By replacing u by $-u$ if necessary, we may assume that $Df(a)u=c>0$. Let $\alpha(t)=a+tu$, choose $\delta_1>0$ small enough so that $\alpha(t)\in U$ for all $|t|<\delta_1$, and let $g(t)=f(\alpha(t))$ for $|t|<\delta_1$. By the Chain Rule we have $g'(t)=Df(\alpha(t))\alpha'(t)$ so that, in particular, $g'(0)=Df(a)u=c>0$. Since $c=g'(0)=\lim\limits_{t\to 0}\frac{g(t)-g(0)}{t}$, we can choose δ with $0<\delta<\delta_1$ such that when $0<|t|<\delta$ we have $\left|\frac{g(t)-g(0)}{t}-c\right|<\frac{c}{2}$, and hence $\frac{c}{2}<\frac{g(t)-g(0)}{t}<\frac{3c}{2}$. For $0<t<\delta$ we have $g(t)-g(0)>\frac{ct}{2}>0$ so that $g(t)>g(0)$. Thus $f(a+tu)>f(a)$ for all $0<t<\delta$, and so f does not have a local maximum at a.

(c) Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_k}{\partial x_\ell}(x)$ exist and are bounded in U. Prove that f is continuous.

Solution: We imitate the proof of Theorem 5.13. Let $\epsilon>0$. Choose $M\geq 0$ so that $\left|\frac{\partial f_k}{\partial x_\ell}(x)\right|\leq M$ for all indices k,ℓ and all $x\in U$ and choose δ with $0<\delta<\frac{\epsilon}{Mnm}$ so that $B(a,\delta)\subseteq U$. Let $x\in B(a,\delta)$. For $0\leq\ell\leq n$, let $u_\ell=(x_1,\cdots,x_\ell,a_{\ell+1},\cdots,a_n)$, with $u_0=a$ and $u_n=x$, and note that each $u_\ell\in B(a,\delta)$. For $1\leq\ell\leq n$, let $\alpha_\ell(t)=(x_1,\cdots,x_{\ell-1},t,a_{\ell+1},\cdots,a_n)$ for t between a_ℓ and x_ℓ. For $1\leq k\leq m$ and $1\leq\ell\leq n$, let $g_{k,\ell}(t)=f_k\big(\alpha_\ell(t)\big)$ so that $g'_{k,\ell}(t)=\frac{\partial f_k}{\partial x_\ell}\big(\alpha_\ell(t)\big)$. By the Mean Value Theorem, we can choose $s_{k,\ell}$ between a_ℓ and x_ℓ so $g'_{k,\ell}(s_{k,\ell})(x_\ell-a_\ell)=g_{k,\ell}(x_\ell)-g_{k,\ell}(a_\ell)$ or, equivalently, so $\frac{\partial f_k}{\partial x_\ell}\big(\alpha_\ell(s_{k,\ell})\big)(x_\ell-a_\ell)=f_k(u_\ell)-f_k(u_{\ell-1})$. Then

$$f_k(x)-f_k(a)=f_k(u_n)-f_k(u_0)=\sum_{\ell=1}^{n}\big(f_k(u_\ell)-f_k(u_{\ell-1})\big)=\sum_{\ell=1}^{n}\frac{\partial f_k}{\partial x_\ell}\big(\alpha_\ell(s_{k,\ell})\big)(x_\ell-a_\ell),$$

so that $\big|f_k(x)-f_k(a)\big|\leq M\sum\limits_{\ell=1}^{n}\big|x_\ell-a_\ell\big|\leq Mn\,|x-a|$. Thus

$$\big|f(x)-f(a)\big|=\Big(\sum_{k=1}^{m}\big|f_k(x)-f_k(a)\big|^2\Big)^{1/2}\leq\Big(\sum_{k=1}^{m}n^2M^2|x-a|^2\Big)^{1/2}=Mnm\,|x-a|<Mnm\,\delta<\epsilon.$$

2: (a) Let $(u,v)=f(x,y)=\left(x\ln(y-x^4),\left(2+\frac{y}{x}\right)^{3/2}\right)$. Explain why f is locally invertible in a neighbourhood of $(1,2)$ and find the linearization of its inverse at $(0,8)$.

Solution: Note that $f(1,2)=(0,8)$. Also

$$DF(x,y)=\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}=\begin{pmatrix}\ln(y-x^4)-\frac{4x^2}{y-x^4} & \frac{x}{y-x^4}\\ -\frac{3y}{2x^2}\left(2+\frac{y}{x}\right)^{1/2} & \frac{3}{2x}\left(2+\frac{y}{x}\right)^{1/2}\end{pmatrix}\text{, so } DF(1,2)=\begin{pmatrix}-4 & 1\\ -6 & 3\end{pmatrix}.$$

F is locally invertible near $(1,2)$ because the matrix $DF(1,2)$ is invertible, and the partial derivatives u_x, u_y, v_x and v_y are all continuous near $(1,2)$. Since $F(1,2)=(0,8)$ we have $F^{-1}(0,8)=(1,2)$, and we have

$$DF^{-1}(0,8)=F(1,2)^{-1}=\begin{pmatrix}-4 & 1\\ -6 & 3\end{pmatrix}^{-1}=\tfrac{1}{6}\begin{pmatrix}3 & -1\\ 6 & -4\end{pmatrix}$$

and so the linearization of F^{-1} at $(0,8)$ is

$$L_{(0,8)}F^{-1}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}+\tfrac{1}{6}\begin{pmatrix}3 & -1\\ 6 & -4\end{pmatrix}\begin{pmatrix}x-0\\ y-8\end{pmatrix}.$$

(b) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=2x^3-3x^2+2y^3+3y^2$ and let $C=\text{Null}(f)$. Use the Implicit Function Theorem to find all the points on C at which C is locally equal to the graph of a function $y=g(x)$, or locally equal to the graph of a function $x=h(y)$.

Solution: By the Implicit Function Theorem, C is locally equal to the graph of a function $y=g(x)$ at all points on C except (possibly) where $\frac{\partial f}{\partial y}=0$, and it is locally equal to the graph of a function $x=h(y)$ at all points on C except (possibly) where $\frac{\partial f}{\partial x}=0$. We have $\frac{\partial f}{\partial y}=6y^2+6y=6y(y+1)$ and so $\frac{\partial f}{\partial y}=0\iff y\in\{0,-1\}$. For $(x,y)\in C$ we have $2x^2-3x^2=-(2y^3+3y^2)$, so

$$y=0\implies 2x^3-3x^2=0\implies x^2(2x-3)=0\implies x\in\left\{0,\tfrac{3}{2}\right\},$$
$$y=-1\implies 2x^3-3x^2=-1\implies 2x^3-3x^2+1=0\implies (x-1)^2(2x+1)=0\implies x\in\left\{1,-\tfrac{1}{2}\right\}.$$

Thus C is locally equal to the graph of a smooth function $y=g(x)$ except (possibly) at the points

$$(x,y)\in\left\{(0,0)\,,\,\left(\tfrac{3}{2},0\right)\,,\,(1,-1)\,,\,\left(-\tfrac{1}{2},-1\right)\right\}.$$

Also, we have $\frac{\partial f}{\partial x}=6x^2-6x=6x(x-1)$ so that $\frac{\partial f}{\partial x}=0\iff x\in\{0,1\}$. When $(x,y)\in C$ so that $2y^3+3y^2=-(2x^3-3x^2)$, we have

$$x=0\implies 2y^3+3y^2=0\implies y^2(2y+3)=0\implies y\in\left\{0,-\tfrac{3}{2}\right\},$$
$$x=1\implies 2y^3+3y^2=1\implies 2y^3+3y^2-1=0\implies (y+1)^2(2y-1)=0\implies y\in\left\{-1,\tfrac{1}{2}\right\}.$$

Thus C is locally equal to the graph of a function $x=h(y)$ at all points in C except (possibly) at the points

$$(x,y)\in\left\{(0,0)\,,\,\left(0,-\tfrac{3}{2}\right)\,,\,(1,-1)\,,\,\left(1,\tfrac{1}{2}\right)\right\}.$$

3: (a) Let $U = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 > y^2\}$. Find the 2nd Taylor polynomial of the map $f : U \to \mathbb{R}$ given by $f(x,y) = \sqrt{x^2 - y^2}$ at the point $(5,4)$.

Solution: We have $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2-y^2}}$, $\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{x^2-y^2}}$, $\frac{\partial^2 f}{\partial x^2} = \frac{\sqrt{x^2-y^2} - \frac{x^2}{\sqrt{x^2-y^2}}}{x^2-y^2} = \frac{-y^2}{(x^2-y^2)^{3/2}}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{xy}{(x^2-yy^2)^{3/2}}$, and $\frac{\partial^2 f}{\partial y^2} = \frac{-\sqrt{x^2-y^2} - \frac{y^2}{\sqrt{x^2-y^2}}}{x^2-y^2} = \frac{-x^2}{(x^2-y^2)^{3/2}}$, so that $f(5,4) = 3$, $\frac{\partial f}{\partial x}(5,4) = \frac{5}{3}$, $\frac{\partial f}{\partial y}(5,4) = -\frac{4}{3}$, $\frac{\partial^2 f}{\partial x^2}(5,4) = -\frac{16}{27}$, $\frac{\partial^2 f}{\partial x \partial y}(5,4) = \frac{20}{27}$ and $\frac{\partial^2 f}{\partial y^2}(5,4) = -\frac{25}{27}$, and hence the 2nd Taylor polynomial of f at $(5,4)$ is

$$T(x,y) = 3 + \tfrac{5}{3}(x-5)_{-}\tfrac{4}{3}(y-4) - \tfrac{8}{27}(x-5)^2 + \tfrac{20}{27}(x-5)(y-4) - \tfrac{25}{54}(y-4)^2.$$

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 - xy + y^3 - y$. Find and classify all the critical points of f in $\mathbb{R}^2$, and find the absolute maximum and minimum values of f on the set $A = \{(x,y) \in \mathbb{R}^2 \,|\, y^2 - 1 \le x \le 2\}$.

Solution: We have $Df(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2x - y\,,\, -x + 3y^2 - 1\right)$, so

$$Df(x,y) = (0,0) \iff \left(y = 2x \text{ and } x = 3y^2 - 1\right) \iff \left(y = 2x \text{ and } x = 3(2x)^2 - 1\right).$$

Since $x = 3(2x)^2 - 1 \iff 12x^2 - x - 1 = 0 \iff (4x+1)(3x-1) = 0 \iff x \in \{\frac{1}{3}, -\frac{1}{4}\}$, we have

$$Df(x,y) = (0,0) \iff (x,y) \in \left\{\left(\tfrac{1}{3}, \tfrac{2}{3}\right), \left(-\tfrac{1}{4}, -\tfrac{1}{2}\right)\right\}.$$

To classify the critical points, we find the Hessian of f. We have

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 6y \end{pmatrix}.$$

At the point $\left(\frac{1}{3}, \frac{2}{3}\right)$ we have $Hf = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 4 \end{smallmatrix}\right)$. The characteristic polynomial is $x^2 - 6x + 7$ so the eigenvalues are $\frac{6 \pm \sqrt{36-28}}{2} = 3 \pm 2$. Since both eigenvalues are positive, f has a local minimum at $\left(\frac{1}{3}, \frac{2}{3}\right)$. At $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ we have $Hf = \left(\begin{smallmatrix} 2 & -1 \\ -1 & -3 \end{smallmatrix}\right)$. The characteristic polynomial is $x^2 + x - 7$ so the eigenvalues are $\frac{-1 \pm \sqrt{29}}{2}$. Since one eigenvalue is positive and the other is negative, f has a saddle point at $\left(-\frac{1}{4}, -\frac{1}{2}\right)$.

Since f is continuous and A is compact, f attains its maximum and minimum values on A. In A^o, f has a local minimum at $\left(\frac{1}{3}, \frac{2}{3}\right)$ with $f\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{9} - \frac{2}{9} + \frac{8}{27} - \frac{2}{3} = -\frac{13}{27}$. The boundary of A is the union of the parabolic curve $x = y^2 - 1$ with $-\sqrt{3} \le y \le \sqrt{3}$ and the line segment $x = 2$ with $-\sqrt{3} \le y \le \sqrt{3}$. When $x = y^2 - 1$ we have $f(x,y) = f(y^2-1, y) = (y^2-1)^2 - (y^2-1)y + y^3 - y = (y^2-1)^2$. For $g(y) = (y^2-1)^2$ we have $g'(y) = 4y(y^2-1)$ so that $g'(y) = 0 \iff y = 0, \pm 1$, and we note that $g(0) = 1$, $g(\pm 1) = 0$ and $g(\pm\sqrt{3}) = 4$ (so the max and min values of f along $x = y^2 - 1$ are $f(2, \pm\sqrt{3}) = 4$ and $f(0, \pm 1) = 0$). When $x = 2$ we have $f(x,y) = f(2,y) = 4 - 2y + y^3 - y = y^3 - 3y + 4$. For $h(y) = y^3 - 3y + 4$ we have $h'(y) = 3y^2 - 3y = 3(y^2 - 1)$ so that $h'(y) = 0 \iff y = \pm 1$, and we note that $h(-\sqrt{3}) = 4$, $h(-1) = 6$, $h(1) = 2$ and $h(\sqrt{3}) = 4$ (so the max and min values of f along $x = 2$ are $f(2,-1) = 6$ and $f(2,1) = 2$). Thus the absolute maximum value of f is $f(2,-1) = 6$ and the absolute minimum value is $f\left(\frac{1}{3}, \frac{2}{3}\right) = -\frac{13}{27}$.

4: (Lagrange Multipliers) For $X \subseteq \mathbb{R}^n$ with $a \in X$, we define the **tangent space** of X at a to be the set $T_a X$ of all vectors $u \in \mathbb{R}^n$ such that there exists $\delta > 0$ and there exists a differentiable map $\alpha : (-\delta, \delta) \subseteq \mathbb{R} \to X \subseteq \mathbb{R}^n$ with $\alpha(0) = a$ such that $\alpha'(0) = u$.

(a) Let $U \subseteq \mathbb{R}^n$ be open with $a \in U$, let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a, let $b = f(a)$. Prove that $T_{(a,b)}\text{Graph}(f) = \text{Graph}(Df(a))$, which is an n-dimensional vector space in $\mathbb{R}^{n+m}$.

Solution: Let $\alpha : (-\delta, \delta) \subseteq \mathbb{R} \to \text{Graph}(f) \subseteq \mathbb{R}^{n+m}$ be differentiable with $\alpha(0) = (a, b)$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Since $\text{Graph}(f) = \big\{(x, y) \,\big|\, x \in U, y = f(x)\big\}$, we can write $\alpha(t) = \big(x(t), y(t)\big) = \big(x(t), f(x(t))\big)$ with $x(t) \in U$ and $x(0) = a$. By the Chain Rule, we have $\alpha'(t) = \big(x'(0), Df(a)x'(0)\big) = \big(u, Df(a)u\big)$ where $u = \alpha'(0) \in \mathbb{R}^n$ and $Df(a)u \in \mathbb{R}^m$. This shows that $T_{(a,b)}\text{Graph}(f) \subseteq \big\{\big(u, Df(a)u\big) \,\big|\, u \in \mathbb{R}^n\big\} = \text{Graph}\big(Df(a)\big)$. On the other hand, given $u \in \mathbb{R}^n$, since U is open with $a \in U$ we can choose $\delta > 0$ so that $a + tu \in U$ for all $t \in (-\delta, \delta)$, then we can define $\alpha : (-\delta, \delta) \to \text{Graph}(f)$ by $\alpha(t) = \big(a + tu, f(a + tu)\big)$ to get $\alpha(0) = (a, b)$ and $\alpha'(0) = (u, Df(a)u)$. This shows that $\text{Graph}\big(Df(a)\big) \subseteq T_{(a,b)}\text{Graph}(f)$. Finally, we recall (see Note 1.23) that $\text{Graph}(Df(a)) = \text{Col}\big(\begin{smallmatrix} I \\ Df(a) \end{smallmatrix}\big)$, which is an n-dimensional vector space.

(b) Let $U \subseteq \mathbb{R}^n$ be open with $a \in U$, let $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{C}^1$ in U with $g(a) = 0$, and suppose that $\text{rank}\big(Dg(a)\big) = m < n$. Prove that $T_a\text{Null}(g) = \text{Null}\big(Dg(a)\big)$, an $(n - m)$-dimensional vector space in $\mathbb{R}^n$.

Solution: Let $\alpha : (-\delta, \delta) \subseteq \mathbb{R} \to \text{Null}(g) \subseteq U \subseteq \mathbb{R}^n$ be differentiable with $\alpha(0) = a$. For all $t \in (-\delta, \delta)$, since $\alpha(t) \in \text{Null}(g)$ we have $g(\alpha(t)) = 0$. By the Chain Rule, $Dg(\alpha(t))\alpha'(t) = 0$. Taking $t = 0$ gives $Dg(a)\alpha'(0) = 0$. This shows that $T_a\text{Null}(g) \subseteq \text{Null}\big(Dg(a)\big)$.

Since g is $\mathcal{C}^1$ in U and $\text{rank}\big(Dg(a)\big) = m$, it follows from the Implicit Function Theorem that $\text{Null}(g)$ is locally equal to the graph of a $\mathcal{C}^1$ function h: reorder the variables in $\mathbb{R}^n$ so that the last m columns of $Dg(a)$ are independent, write elements in $\mathbb{R}^n$ as (x, y) with $x \in \mathbb{R}^{n-m}$ and $y \in \mathbb{R}^m$ and write $a = (b, c)$, choose an open set $U_0 \subseteq \mathbb{R}^n$ with $a = (b, c) \in U_0 \subseteq U$, and a $\mathcal{C}^1$ function $h : W_0 \subseteq \mathbb{R}^{n-m} \to \mathbb{R}^m$ with $h(b) = c$ so that $\text{Null}(g) \cap U_0 = \text{Graph}(h)$. Because these two sets are equal, it follows that $T_a\text{Null}(g) = T_{(b,c)}\text{Graph}(h)$ $\big($indeed, given a differentiable map $\alpha : (-\delta, \delta) \to \text{Null}(g)$ with $\alpha(0) = a$, we can restrict the domain by choosing δ_0 with $0 < \delta_0 \leq \delta$ so that $\alpha : (-\delta_0, \delta_0) \to \text{Null}(g) \cap U_0\big)$. Since $T_a\text{Null}(g) = T_{(b,c)}\text{Graph}(h)$, we know from Part (a) that $T_a\text{Null}(g)$ is an $(n - m)$-dimensional vector space. Since $T_a\text{Null}(g)$ and $\text{Null}\big(Dg(a)\big)$ are both $(n - m)$-dimensional vector spaces with $T_a\text{Null}(g) \subseteq \text{Null}\big(Dg(a)\big)$, they must be equal.

(c) Let $U \subseteq \mathbb{R}^n$ be open with $a \in U$, let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at a, let $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{C}^1$ in U with $g(a) = 0$ and $\text{rank}\big(Dg(a)\big) = m < n$. Prove that if $f(a) \geq f(x)$ for all $x \in \text{Null}(g)$, or if $f(a) \leq f(x)$ for all $x \in \text{Null}(g)$, then $\nabla f(a) \in \text{Row}\big(Dg(a)\big) = \text{Span}\big\{\nabla g_1(a), \cdots, \nabla g_m(a)\big\}$.

Solution: Let $u \in \text{Null}(Dg(a))$. By Part (b), we have $u \in T_a\text{Null}(g)$, so we can choose a differentiable map $\alpha : (-\delta, \delta) \subseteq \mathbb{R} \to \text{Null}(g)$ with $\alpha(0) = a$ and $\alpha'(0) = u$. Define $h : (-\delta, \delta) \subseteq \mathbb{R} \to \mathbb{R}$ by $h(t) = f(\alpha(t))$. By the Chain Rule, h is differentiable in $(-\delta, \delta)$ with $h'(t) = Df(\alpha(t))\alpha'(t)$ for all $t \in (-\delta, \delta)$. From our assumption that either $f(a) \geq f(x)$ for all $x \in \text{Null}(g)$ or $f(a) \leq f(x)$ for all $x \in \text{Null}(g)$, it follows that either $h(0) \geq h(t)$ for all $t \in (-\delta, \delta)$ or $h(0) \leq h(t)$ for all $t \in (-\delta, \delta)$ and, in either case, we must have $h'(0) = 0$. Thus $0 = h'(0) = Df(\alpha(0))\alpha'(0) = Df(a)u = \nabla f(a) \cdot u$. Since $\nabla f(a) \cdot u = 0$ for every $u \in \text{Null}(Dg(a))$, we have $\nabla f(a) \in \big(\text{Null}(Dg(a))\big)^\perp = \text{Row}(Dg(a))$.

(d) Using Part (c), find the maximum and minimum values of $f(x,y,z)=xy$ on the circle in $\mathbb{R}^3$ given by $x^2+y^2+z^2=1$ and $x+y+z=0$.

Solution: Define $g:\mathbb{R}^3\to\mathbb{R}^2$ by $g(x,y,z)=\big(x^2+y^2+z^2-1\,,\,x+y+z\big)$ so that the given circle is $C=\text{Null}(g)$. We remark that C is compact, so f attains its maximum and minimum values on C. We have

$$Dg(x,y,z)=\begin{pmatrix}2x&2y&2z\\1&1&1\end{pmatrix}\ \text{ and }\ \nabla f(x,y,z)=\begin{pmatrix}y\\x\\0\end{pmatrix}.$$

By Part (c), if f attains a maximum or minimum value at $(x,y,z)\in C$, then $\nabla f(x,y,z)\in\text{Row}Dg(x,y,z)$, that is $(y,x,0)=s(2x,2y,2z)+t(1,1,1)$ for some $s,t\in\mathbb{R}$. We solve the equations $y=2sx+t$ (1), $x=2sy+t$ (2), $0=2sz+t$ (3) along with $x^2+y^2+z^2=1$ (4) and $x+y+z=0$ (5). Subtract (2) from (1) to get $y-x=2s(x-y)$, that is $(2s+1)(x-y)=0$ so that either $s=\frac{1}{2}$ or $x=y$. First consider the case that $s=\frac{1}{2}$. Equation (3) gives $-z+t=0$ so that $t=z$, and putting $s=-\frac{1}{2}$ and $t=z$ into (1) gives $x=-y+z$ so that $x+y=z$. Putting $x+y=z$ into (5) gives $2z=0$ so that $z=0$, hence also $x+y=0$ so that $y=-x$. Then putting $y=-x$ and $z=0$ into (4) gives $2x^2=1$ so that $x=\pm\frac{1}{\sqrt{2}}$, Thus in the case $s=\frac{1}{2}$ we obtain the solutions $(x,y,z)=\pm\frac{1}{\sqrt{2}}(1,-1,0)$. Now consider the case that $x=y$. Putting $y=x$ into (5) gives $2x+z=0$ so that $z=-2x$, then putting $y=x$ and $z=-2x$ into (4) gives $6x^2=1$ so that $x=\pm\frac{1}{\sqrt{6}}$. Thus in the case $x=y$ we obtain the solutions $(x,y,z)=\pm\frac{1}{\sqrt{6}}(1,1,-2)$. Thus the maximum value of f on C is $f\big(\pm\frac{1}{\sqrt{6}}(1,1,-2)\big)=\frac{1}{6}$ and the minimum value is $f\big(\pm\frac{1}{\sqrt{2}}(1,-1,0)\big)=-\frac{1}{2}$.