Chapter 1. Introduction to Vector Valued Functions

In this chapter we provide an informal introduction to vector-valued functions of several variables. We describe several ways of associating a geometric object, such as a curve or surface, to a given function. Alternatively, given a geometric object, we describe how to represent the object analytically, that is by using equations, in various ways.

1.1 Definition: Let $D \subseteq \mathbb{R}^n$. We say that f is a **function** or a **map** from D to $\mathbb{R}^m$, and we write $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, when for every $x \in D$ there is a unique point $y = f(x) \in \mathbb{R}^m$. The set D is called the **domain** of the function f.

The **graph** of the function f is the set

$$\text{Graph}(f) = \big\{(x, f(x))\big|x \in D\big\} \subseteq \mathbb{R}^{n+m}.$$

We say the graph of f is defined **explicitly** by the equation $y = f(x)$.

The **null set** of f is the set

$$\text{Null}(f) = f^{-1}(0) = \big\{x \in D\big|f(x) = 0\big\} \subseteq \mathbb{R}^n.$$

More generally, given $k \in \mathbb{R}^m$, the **level set** $f^{-1}(k)$, also called the **inverse image** of k under f, is the set

$$f^{-1}(k) = \big\{x \in D\big|f(x) = k\big\} \subseteq \mathbb{R}^n.$$

More generally still, given a subset $B \subseteq \mathbb{R}^n$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \big\{x \in D\big|f(x) \in B\big\} \subseteq \mathbb{R}^n.$$

We say the level set $f^{-1}(k)$ is defined **implicitly** by the equation $f(x) = k$.

The **range** of f, also called the **image** of f, is the set

$$\text{Range}(f) = f(D) = \big\{f(x))\big|x \in D\big\} \subseteq \mathbb{R}^m.$$

More generally, given a set $A \subseteq D$, the **image** of A under f is the set

$$f(A) = \big\{f(x)\big|x \in A\big\} \subseteq \mathbb{R}^m.$$

We say the range of f is defined **parametrically** by the equation $y = f(x)$, and for $x = (x_1, x_2, \cdots, x_n) \in D$, the variables $x_1, x_2, \cdots, x_n$ are called the **parameters**.

1.2 Note: The graph, the level sets and the range of a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are geometric objects such as points, curves, surfaces, or higher dimensional analogues of these. In accordance with the above definitions, a curve in $\mathbb{R}^2$ or in $\mathbb{R}^3$, or a surface in $\mathbb{R}^3$, can be defined explicitly, implicitly, or parametrically.

A curve in $\mathbb{R}^2$ can be defined explicitly as the graph of a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$, implicitly as the null set (or a level set) of a function $g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, or parametrically as the range of a function $\alpha : D \subseteq \mathbb{R} \to \mathbb{R}^2$.

A curve in $\mathbb{R}^3$ can be defined explicitly as the graph of a function $f : D \subseteq \mathbb{R} \to \mathbb{R}^2$, implicitly as the null set (or a level set) of a function $g : D \subseteq \mathbb{R}^3 \to \mathbb{R}^2$, or parametrically as the range of a map $\alpha : D \subseteq \mathbb{R} \to \mathbb{R}^3$.

A surface in $\mathbb{R}^3$ can be defined explicitly as the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$, implicitly as the null set (or as a level set) of a function $g : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, or parametrically as the range of a function $\sigma : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

1.3 Example: Consider the unit circle $x^2+y^2=1$ in $\mathbb{R}^2$. For $f:[-1,1]\subseteq\mathbb{R}\to\mathbb{R}$ given by $f(x)=\sqrt{1-x^2}$, the graph of f, that is the curve $y=f(x)$, is equal to the top half of the unit circle. For $g:\mathbb{R}^2\to\mathbb{R}$ given by $g(x,y)=x^2+y^2-1$, the null set of g, that is the curve $x^2+y^2=1$, is equal to the entire circle. For $\alpha:\mathbb{R}\to\mathbb{R}^2$ given by $\alpha(t)=(\cos t,\sin t)$, the range of α, that is the curve $(x,y)=\alpha(t)$, is equal to the entire circle.

1.4 Example: Consider the ellipse which is the intersection of the cylinder $x^2+y^2=1$ with the plane $z=x+y$ in $\mathbb{R}^3$. The ellipse is given implicitly by the two equations $x^2+y^2=1$ and $z=x+y$, which can be written in vector form as the single equation $\big(x^2+y^2-1\,,\,z-x-y\big)=(0,0)$, and so it is the null set of the function $g:\mathbb{R}^3\to\mathbb{R}^2$ given by $g(x,y,z)=\big(x^2+y^2-1\,,\,z-x-y\big)$. To obtain a parametric description of the ellipse, note that to get $x^2+y^2=1$ we can take $x=\cos t$ and $y=\sin t$, and then to get $z=x+y$ we can take $z=\cos t+\sin t$, and so the ellipse is given parametrically by $(x,y,z)=\big(\cos t\,,\,\sin t\,,\,\cos t+\sin t\big)$. In other words, the ellipse is the range of the function $\alpha:\mathbb{R}\to\mathbb{R}^3$ given by $\alpha(t)=\big(\cos t\,,\,\sin t\,,\,\cos t+\sin t\big)$. To obtain an explicit description for half of the ellipse, note that the top half of the circle $x^2+y^2=1$ is given by $y=\sqrt{1-x^2}$ and then to get $z=x+y$ we need $z=x+\sqrt{1-x^2}$, and so the right half of the ellipse (when the y-axis points to the right) is given explicitly by $(y,z)=\big(\sqrt{1-x^2}\,,\,x+\sqrt{1-x^2}\big)$. In other words, the right half of the ellipse is the graph of the function $g:[-1,1]\subseteq\mathbb{R}\to\mathbb{R}^2$ given by $g(x)=\big(\sqrt{1-x^2}\,,\,x+\sqrt{1-x^2}\big)$.

1.5 Example: Consider the unit sphere in $\mathbb{R}^3$ given by $x^2+y^2+z^2=1$. The top half of the sphere is the graph $z=f(x,y)$ where $f:D\subseteq\mathbb{R}^2\to\mathbb{R}$ is given by $f(x,y)=\sqrt{1-x^2-y^2}$ with $D=\big\{(x,y)\in\mathbb{R}^2\big|x^2+y^2\leq 1\big\}$. The entire sphere is the null set $g(x,y,z)=0$ where $g:\mathbb{R}^3\to\mathbb{R}$ is given by $g(x,y,z)=x^2+y^2+z^2-1$. The top half of the sphere can be given parametrically by $x=r\cos\theta$ and $y=r\sin\theta$ and $z=\sqrt{1-r^2}$, so it is the range $(x,y,z)=\sigma(r,\theta)$ where $\sigma:D\subseteq\mathbb{R}^2\to\mathbb{R}^3$ is given by $\sigma(r,\theta)=\big(r\cos t\,,\,r\sin t\,,\,\sqrt{1-r^2}\big)$ with $D=\big\{(r,\theta)\in\mathbb{R}^2\big|0\leq r\leq 1\big\}$.

1.6 Remark: A function is uniquely determined by its graph but not by its null set or by its image. It follows that implicit and parametric descriptions of curves and surfaces are not unique. For example, the parabola $y=x^2$ can be given implicitly by $g(x,y)=0$ for any of the functions $g(x,y)=y-x^2$, $g(x,y)=(y-x^2)^3$ or $g(x,y)=(y-x^2)(x^2+1)$, and it can be given parametrically by $(x,y)=\alpha(t)$ for any of the functions $\alpha(t)=(t,t^2)$, $\alpha(t)=(t^3,t^6)$ or $\alpha(t)=(\sinh t\,,\,\sinh^2 t)$.

1.7 Remark: Given an explicit equation for a curve or surface it is easy to obtain an implicit or parametric equation for the curve or surface. For example the curve $y=f(x)$ in $\mathbb{R}^2$ can be given implicitly by $g(x,y)=0$ where $g(x,y)=y-f(x)$ and parametrically by $(x,y)=\alpha(t)$ where $\alpha(t)=\big(t,f(t)\big)$, Similarly the surface $z=f(x,y)$ in $\mathbb{R}^3$ can be given implicitly by $g(x,y,z)=0$ where $g(x,y,z)=z-f(x,y)$ and parametrically by $(x,y,z)=\sigma(s,t)$ where $\sigma(s,t)=\big(s,t,f(s,t)\big)$. On the other hand, given an implicit or a parametric equation for a curve or a surface it can be difficult or impossible to obtain an explicit equation.

1.8 Exercise: The **helix** is given explicitly by $x=\cos z$ and $y=\sin z$. Sketch the curve and find an implicit and a parametric equation for the curve.

1.9 Exercise: The **alpha curve** is given implicitly by $y^2 = x^3 + x^2$. Sketch the curve, find explicit equations for the top and bottom halves of the curve, and find a parametric equation for the entire curve.

1.10 Exercise: The curve which is given explicitly in polar coordinates by $r = r(\theta)$ is given parametrically in Cartesian coordinates by $(x, y) = \alpha(t) = \big(r(t)\cos t\,,\, r(t)\sin t\big)$. Sketch the **cardioid** which is given in polar coordinates by $r = r(\theta) = 1 + \cos\theta$, then find an implicit equation for the curve.

1.11 Exercise: The **twisted cubic** is given parametrically by $(x, y, z) = \alpha(t) = (t, t^2, t^3)$. Sketch the curve and find an implicit and an explicit equation for the curve.

1.12 Remark: To sketch a surface which is defined explicitly as a graph $z = f(x, y)$ or implicitly as a level set $g(x, y, z) = k$, it often helps to first sketch curves of intersection of the surface with various planes $x = c$, $y = c$ or $z = c$. These curves of intersection are also sometimes called **level sets**. The intersection of the graph $z = f(x, y)$ with the plane $z = c$ is given implicitly by $f(x, y) = c$. The intersection of the level set $g(x, y, z) = k$ with the plane $z = c$ is given implicitly by $g(x, y, c) = k$

1.13 Exercise: Sketch the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the parabolic sheet $z = x^2$ and find implicit, explicit, and parametric equations for the curve.

1.14 Exercise: Sketch the surface $z = x^2 + y^2$.

1.15 Exercise: Sketch the surface $z = 4x^2 - y^2$.

1.16 Exercise: Sketch the surface $x^2 + 4y^2 - z^2 = 0$.

1.17 Exercise: Sketch the surface $(x, y, z) = \sigma(u, v) = \big(u, v, u^2 + 4v^2 - 3\big)$.

1.18 Exercise: Find a parametric equation $(x, y, z) = \sigma(\phi, \theta)$ for the sphere of radius r centred at the origin, where the parameters ϕ and θ are the angles of latitude and longitude. In other words, find $\sigma(\phi, \theta)$ so that when $(x, y, z) = \sigma(\phi, \theta)$, ϕ is the angle between $(0, 0, 1)$ and (x, y, z) and θ is the angle from $(1, 0)$ counterclockwise to (x, y).

1.19 Exercise: Find implicit and parametric equations for the **torus** which is obtained by rotating the circle $(x, z) = \big(R + r\cos\theta\,,\, r\sin\theta\big)$ about the z-axis.

1.20 Definition: An **affine space** in $\mathbb{R}^n$ is a set of the form $p + V = \{p + v | v \in V\}$ for some $p \in \mathbb{R}^n$ and some vector space $V \subseteq \mathbb{R}^n$. The **dimension** of the affine space $p + V$ is the same as the dimension of V. The set $p + V$ is called the affine space through p parallel to V, or the affine space through p perpendicular to $V^\perp$, where $V^\perp$ is the **orthogonal complement** of V, given by $V^\perp = \big\{x \in \mathbb{R}^n \,\big|\, x \cdot v = 0 \text{ for all } v \in V\big\}$.

1.21 Example: In $\mathbb{R}^3$, the only zero dimensional vector space is the origin $\{0\}$, the 1-dimensional vector spaces are the lines through the origin, the 2-dimensional spaces are the planes through the origin, and the only 3-dimensional vector space is all of $\mathbb{R}^3$. The 0-dimensional affine spaces are the points in $\mathbb{R}^3$, the 1-dimensional affine spaces are the lines in $\mathbb{R}^3$, the 2-dimensional affine spaces are the planes in $\mathbb{R}^3$, and the only 3-dimensional affine space is all of $\mathbb{R}^3$.

1.22 Definition: Let $f : \mathbb{R}^n \to \mathbb{R}^m$. The function f is called **linear** when it is of the form $f(x) = Ax$ for some matrix $A \in M_{m\times n}(\mathbb{R})$, and f is called **affine** when it is of the form $f(x) = Ax + b$ for some matrix $A \in M_{m\times n}(\mathbb{R})$ and some vector $b \in \mathbb{R}^m$.

1.23 Note: Let $A \in M_{m\times n}$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map $f(x) = Ax$. Let $u_1, \cdots, u_n \in \mathbb{R}^m$ be the column vectors of A and let $v_1, \cdots, v_m \in \mathbb{R}^n$ be the row vectors of A so that we have $A = \big(u_1, \cdots, u_n\big) = \big(v_1, \cdots, v_m\big)^T$. Let $c \in \mathbb{R}^m$ be a point in the range of f, say $f(p) = c$ where $p \in \mathbb{R}^n$. Then

$$\begin{aligned}
\text{Range}(f) &= \big\{Ax \big| x \in \mathbb{R}^n\big\} = \Big\{ \sum_{i=1}^n u_i x_i \Big| \text{ each } x_i \in \mathbb{R}\Big\} = \text{Span}\{u_1, \cdots, u_n\} = \text{Col}(A), \\
\text{Null}(f) &= \text{Null}(A) = \big\{x \in \mathbb{R}^n \big| Ax = 0\big\} = \big\{x \in \mathbb{R}^n \big| v_i \cdot x = 0 \text{ for all } i\big\} = \text{Row}(A)^\perp, \\
f^{-1}(c) &= \big\{x \in \mathbb{R}^n \big| Ax = c\big\} = \big\{x \in \mathbb{R}^n \big| Ax = Ap\big\} = \big\{x \in \mathbb{R}^n \big| A(x-p) = 0\big\} \\
&= \big\{p + y \in \mathbb{R}^n \big| Ay = 0\big\} = p + \text{Null}(A) \text{ , and} \\
\text{Graph}(f) &= \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix} \middle| x \in \mathbb{R}^n \right\} = \text{Span}\left\{ \begin{pmatrix} e_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ u_2 \end{pmatrix}, \cdots, \begin{pmatrix} e_n \\ u_n \end{pmatrix} \right\} = \text{Col}\begin{pmatrix} I \\ A \end{pmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\dim\big(\text{Graph}(f)\big) = n \\
&\dim\big(\text{Range}(f)\big) = \text{rank}(A) \text{ and} \\
&\dim\big(\text{Null}(f)\big) = \dim\big(f^{-1}(c)\big) = \text{nullity}(A) = n - \text{rank}(A).
\end{aligned}$$

1.24 Note: Let $A \in M_{m\times n}(\mathbb{R})$, let $b \in \mathbb{R}^m$ and let $f(x) = Ax + b$. Let $c \in \mathbb{R}^m$ be in the range of f with say $f(p) = c$ where $p \in \mathbb{R}^n$. Then

$$\begin{aligned}
\text{Graph}(f) &= \left\{ \begin{pmatrix} x \\ Ax+b \end{pmatrix} \middle| x \in \mathbb{R}^n \right\} = \begin{pmatrix} 0 \\ b \end{pmatrix} + \text{Col}\begin{pmatrix} I \\ A \end{pmatrix}, \\
\text{Range}(f) &= \big\{Ax + b \big| x \in \mathbb{R}^n\big\} = b + \text{Col}(A) \text{ , and} \\
f^{-1}(c) &= \big\{x \in \mathbb{R}^n \big| Ax + b = c = Ap + b\big\} = \big\{x \in \mathbb{R}^n \big| A(x-p) = 0\big\} = p + \text{Null}(A),
\end{aligned}$$

Note that if $u_1, u_2, \cdots, u_n \in \mathbb{R}^m$ are the columns of A and $e_1, e_2, \cdots, e_n \in \mathbb{R}^n$ are the standard basis vectors for $\mathbb{R}^n$, then we have $f(0) = b$ and $f(e_i) = Ae_i + b = u_i + b$. If $v_1, \cdots, v_m \in \mathbb{R}^n$ are the row vectors of A, so $A = (v_1, \cdots, v_m)^T$, and $k = c - b$, then since

$$f(x) = c \iff Ax + b = c \iff Ax = k \iff v_i \cdot x = k_i \text{ for all } i,$$

it follows that the level set $f(x) = c$ is the intersection of the affine spaces $v_i \cdot x = k_i$, and we note that the space $v_i \cdot x = k_i$ is the affine space in $\mathbb{R}^n$ of dimension $n - 1$ through p perpendicular to v_i.

1.25 Exercise: Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = \big(x + 3y + 2z\,,\, 2z + 5y + 3z\big)$ and let $(a, b) = (1, 1)$. Find a parametric equation for the level set $f(x, y, z) = (a, b)$.

1.26 Exercise: Let $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 5 & 2 & -4 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and let $f(x) = Ax + b$. Find an implicit equation for the range of f.