Chapter 3. Limits and Continuity

3.1 Definition: For $p \in \mathbb{Z}$, let $\mathbb{Z}_{\geq p} = \big\{n \in \mathbb{Z} | n \geq p\big\} = \{p, p+1, p+2, \cdots\}$. For a set A, a **sequence** in A is a function $a : \mathbb{Z}_{\geq p} \to A$ for some $p \in \mathbb{Z}$. We write $(a_n)_{n\geq p}$ to denote the sequence $a : \mathbb{Z}_{\geq p} \to A$ given by $a(n) = a_n$, where $a_n \in A$ for all $n \geq p$. A **subsequence** of the sequence $(a_n)_{n\geq p}$ is a sequence of the form $(b_k)_{k\geq q}$ with $b_k = a_{n_k}$ for some $p \leq n_k < n_{k+1}$ for all $k \geq q$.

3.2 Definition: Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. We say the sequence $(a_n)_{n\geq p}$ is **bounded** when

$$\exists r > 0 \ \forall n \in \mathbb{Z}_{\geq p} \ |a_n| \leq r.$$

For $b \in \mathbb{R}^m$, we say that the sequence $(a_n)_{n\geq p}$ **converges to** b and write $\lim\limits_{n\to\infty} a_n = b$ (or $a_n \to b$) when

$$\forall \epsilon > 0 \ \exists N \in \mathbb{Z}_{\geq p} \ \forall n \in \mathbb{Z}_{\geq p} \big(n \geq N \implies |a_n - b| < \epsilon\big).$$

We say the sequence $(a_n)_{n\geq p}$ **diverges to** ∞ and write $\lim\limits_{n\to\infty} a_n = \infty$ (or $a_n \to \infty$) when

$$\forall r > 0 \ \exists N \in \mathbb{Z}_{\geq p} \ \forall n \in Z_{\geq p} \big(n \geq N \implies |a_n| \geq r\big).$$

We say that the sequence $(a_n)_{n\geq p}$ **converges** when it converges to some point $b \in \mathbb{R}^m$ and otherwise we say that it **diverges**.

3.3 Theorem: *(Convergent Sequences are Bounded) Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. If $(a_n)_{n\geq p}$ converges in $\mathbb{R}^m$ then $(a_n)_{n\geq p}$ is bounded.*

Proof: Suppose that $(a_n)_{n\geq p}$ converges in $\mathbb{R}^m$. Let $u = \lim\limits_{n\to\infty} a_n \in \mathbb{R}^m$. Choose $N \geq p$ such that $n \geq N \implies |a_n - u| < 1$. For $n \geq N$, by the Triangle Inequality we have $|a_n| \leq |a_n - u| + |u| < 1 + |u|$. Thus we can choose $r = \max\big\{|a_p|, |a_{p+1}|, \cdots, |a_{N-1}|, 1+|u|\big\}$ to obtain $|a_n| \leq r$ for all $n \geq p$, and so the sequence $(a_n)_{n\geq p}$ is bounded, as required.

3.4 Theorem: *(Uniqueness of Limits of Sequences) Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$ and let $u, v \in \mathbb{R}^m \cup \{\infty\}$. If $\lim\limits_{n\to\infty} a_n = u$ and $\lim\limits_{n\to\infty} a_n = v$ then $u = v$.*

Proof: We prove the theorem in the case that $u, v \in \mathbb{R}^m$ and leave the case that $u = \infty$ or $v = \infty$ as an exercise. Suppose that $\lim\limits_{n\to\infty} a_n = u \in \mathbb{R}^m$ and $\lim\limits_{n\to\infty} a_n = v \in \mathbb{R}^m$. Suppose, for a contradiction, that $u \neq v$. Choose $N_1 \geq p$ such that $n \geq N_1 \implies |a_n - u| < \frac{|u-v|}{2}$ and choose $N_2 \geq p$ such that $n \geq N_2 \implies |a_n - v| < \frac{|u-v|}{2}$. Let $N = \max\{N_1, N_2\}$. For $n \geq N$ we have $|u - v| \leq |u - a| + |a - v| < \frac{|u-v|}{2} + \frac{|u-v|}{2} = |u - v|$ which is impossible. Thus we must have $u = v$, as required.

3.5 Theorem: *(Limits of Subsequences) Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$ and let $(a_{n_k})_{k\geq q}$ be a subsequence of $(a_n)_{n\geq p}$. If $\lim\limits_{n\to\infty} a_n = u \in \mathbb{R}^m \cup \{\infty\}$ then $\lim\limits_{k\to\infty} a_{n_k} = u$.*

Proof: We give the proof in the case that $u \in \mathbb{R}^m$. Suppose that $\lim\limits_{n\to\infty} a_n = u \in \mathbb{R}^m$ and let $(a_{n_k})_{k\geq q}$ be any subsequence of (a_n). Let $\epsilon > 0$. Choose $N \geq p$ such that $n \geq N \implies |a_n - u| < \epsilon$. Choose $M \geq q$ such that $k \geq M \implies n_k \geq N$ (we can do this since each $n_k \in \mathbb{Z}$ with $n_k < n_{k+1}$ and hence $n_k \to \infty$ as $k \to \infty$). Then for $k \geq M$ we have $n_k \geq N$ and so $|a_{n_k} - u| < \epsilon$. Thus $\lim\limits_{k\to\infty} a_{n_k} = u$, as required.

3.6 Remark: It follows from the above theorem that the initial index p of a sequence $(a_n)_{n\geq p}$ does not effect whether or not the sequence converges, and it does not influence the value of the limit. For this reason, we often omit the initial index p from our notation and denote the sequence $(a_n)_{n\geq p}$ simply as (a_n).

3.7 Definition: Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. For $n\geq p$ let $a_n=(a_{n,1},a_{n,2},\cdots,a_{n,m})$. For each index k with $1\leq k\leq m$, the k^{th} **component sequence** of $(a_n)_{n\geq p}$ is the sequence $(a_{n,k})_{n\geq p}=(a_{p,k},a_{p+1,k}\cdots)$. Note that the sequence $(a_n)_{n\geq p}$ in $\mathbb{R}^m$ determines and is determined by its component sequences $(a_{n,k})_{n\geq p}$.

3.8 Theorem: *(Limits of Component Sequences) Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$, say $a_n=\big(a_{n,1},a_{n,2},\cdots,a_{n,m}\big)\in\mathbb{R}^m$.*
(1) $(a_n)_{n\geq p}$ is bounded if and only if $(a_{n,k})_{n\geq p}$ is bounded for all indices k.
(2) For $b=(b_1,\cdots,b_m)\in\mathbb{R}^m$ we have $\lim\limits_{n\to\infty}a_n=b$ if and only if $\lim\limits_{n\to\infty}a_{n,k}=b_k$ for all k.

Proof: Suppose that $(a_n)_{n\geq p}$ is bounded. Choose $r>0$ such that $|a_n|\leq r$ for all $n\geq p$. Let $n\geq p$ and let $1\leq k\leq m$. Then $|a_{n,k}|\leq|a_n|\leq r$ and so the sequence $(a_{n,k})_{n\geq p}$ is also bounded. Now suppose, conversely, that $(a_{n,k})_{n\geq p}$ is bounded for all indices k. For each k, chose $r_k>0$ such that $|a_{n,k}|\leq r_k$ for all $n\geq p$. Let $r=r_1+\cdots+r_m$. Then for all $n\geq p$, by the Triangle Inequality we have $|a_n|\leq|a_{n,1}|+|a_{n,2}|+\cdots+|a_{n,m}|<r_1+r_2+\cdots+r_m=r$ and so the sequence $(a_n)_{n\geq p}$ is bounded. This proves Part (1).

To prove Part (2), suppose first that $\lim\limits_{n\to\infty}a_n=b$. Let $\epsilon>0$ and choose $N\geq p$ so that $n\geq N\Longrightarrow\big|a_n-b\big|<\epsilon$. Let $1\leq k\leq m$. For $n\geq N$ we have $\big|a_{n,k}-b_k\big|\leq\big|a_n-b\big|<\epsilon$ and so $\lim\limits_{n\to\infty}a_{n,k}=b_k$. Now suppose, conversely, that $\lim\limits_{n\to\infty}a_{n,k}=b_k$ for all indices k. Let $\epsilon>0$. For each index k, choose $N_k\geq p$ such that $n\geq N_k\Longrightarrow\big|a_{n,k}-b_k\big|<\frac{\epsilon}{m}$. Then for $n\geq N$, by the Triangle Inequality we have $|a_n-b|\leq\sum\limits_{k=1}^{m}|a_{n,k}-b_k|<\epsilon$ and so $\lim\limits_{n\to\infty}a_n=b$.

3.9 Theorem: *(Operations on Limits of Sequences) Let (a_n) and (b_n) be sequences in $\mathbb{R}^m$ and let $c\in\mathbb{R}$. Suppose that $\lim\limits_{n\to\infty}a_n=u\in\mathbb{R}^m$ and $\lim\limits_{n\to\infty}b_n=v\in\mathbb{R}^m$. Then*

(1) $\lim\limits_{n\to\infty}(a_n+b_n)=u+v$,
(2) $\lim\limits_{n\to\infty}(c\,a_n)=c\,u$,
(3) $\lim\limits_{n\to\infty}|a_n|=|u|$,
(4) $\lim\limits_{n\to\infty}(a_n\cdot b_n)=u\cdot v$, and
(5) if $m=3$ then $\lim\limits_{n\to\infty}(a_n\times b_n)=u\times v$.

Proof: These follow easily from Part (2) of the above theorem and from known properties of sequences in $\mathbb{R}$. For eample, to prove Part (1), note that

$$\lim_{n\to\infty}(a_n+b_n)_k=\lim_{n\to\infty}(a_{n,k}+b_{n,k})=\lim_{n\to\infty}a_{n,k}+\lim_{n\to\infty}b_{n,k}=u_k+v_k=(u+v)_k.$$

3.10 Theorem: *(Sequential Characterization of Limit Points) Let $A \subseteq \mathbb{R}^m$ and let $a \in \mathbb{R}^m$. Then $a \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{a\}$ such that $\lim\limits_{n\to\infty} x_n = a$.*

Proof: Let $a \in A'$. For each $n \in \mathbb{Z}^+$, since $a \in A'$ we have $B^*\big(a, \frac{1}{n}\big) \cap A \neq \emptyset$ so we can choose an element $x_n \in B^*\big(a, \frac{1}{n}\big) \cap A$ and then we have $x_n \in A \setminus \{a\}$ and $|x_n - a| < \frac{1}{n}$. Given $\epsilon > 0$ we can choose a positive integer $N > \frac{1}{\epsilon}$ and then we have $n \geq N \Longrightarrow |x_n - a| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Thus $(x_n)_{n\geq 1}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$.

Suppose, conversely, that $(x_n)_{n\geq p}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$. Let $r > 0$. Since $\lim\limits_{n\to\infty} x_n = a$ we can choose $N \geq p$ so that $n \geq N \Longrightarrow |x_n - a| < r$. Then we have $x_N \in A \setminus A$ and $|x_N - a| < r$ so that $x_N \in B^*\big(a, r\big)$, and hence $B^*\big(a, r\big) \neq 0$. Since $r > 0$ was arbitrary, it follows that $a \in A'$.

3.11 Theorem: *(Sequential Characterization of Closed Sets) Let $A \subseteq \mathbb{R}^m$. Then A is closed (in $\mathbb{R}^m$) if and only if every for every sequence in A which converges in $\mathbb{R}^m$, the limit of the sequence lies in A.*

Proof: Suppose that A is closed. Let $(x_n)_{n\geq p}$ be a sequence in A which converges in $\mathbb{R}^n$. Let $a = \lim\limits_{n\to\infty} x_n$. Suppose, for a contradiction, that $a \notin A$. Since $a \notin A$ we have $A = A \setminus \{a\}$ and so (x_n) is a sequence in $A \setminus \{a\}$. Since (x_n) is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$, we have $a \in A'$ by the Characterization of Limit Points. Since A is closed we have $A' \subseteq A$ and so $a \in A$, giving the desired contradiction.

Suppose, conversely, that for every sequence in A which converges in $\mathbb{R}^n$, the limit of the sequence lies in A. Let $a \in A'$. By the Characterization of Limit Points, we can choose a sequence (x_n) in $A \setminus \{a\}$ such that $\lim\limits_{n\to\infty} x_n = a$. Then (x_n) is a sequence in A which converges in $\mathbb{R}^n$, and so its limit must lie in A, thus we have $a \in A$. Since $a \in A'$ was arbitrary, this proves that $A' \subseteq A$ and so A is closed.

3.12 Theorem: *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.*

Proof: For this proof, we shall label the components of an element in $\mathbb{R}^m$ using superscripts rather than subscripts, so we shall write an element $x \in \mathbb{R}^m$ as $(x^1, x^2, \cdots, x^m)$. Let (x_n) be a bounded sequence in $\mathbb{R}^m$. Then the first component sequence (x_n^1) is a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass Theorem for sequences in $\mathbb{R}$, we can choose a convergent subsequence $(x_{n_\ell}^1)$, where $n_1 < n_2 < \cdots$. Since the second component sequence (x_n^2) is bounded, the subsequence $(x_{n_\ell}^2)$ is also bounded so we can choose a convergent subsequence $(x_{n_{\ell_k}}^2)$, where $\ell_1 < \ell_2 < \cdots$. Note that the sequence $(x_{n_{\ell_k}}^1)$ also converges because it is a subsequence of the convergent subsequence $(x_{n_\ell}^1)$. Since the sequence (x_n^3) is bounded, the subsequence $(x_{n_{\ell_k}}^3)$ is also bounded so we can choose a convergent subsequence $(x_{n_{\ell_{k_j}}}^3)$, where $k_1 < k_2 < \cdots$. We then obtain convergent subsequences of each of the first 3 component sequences (x_n^i) for $i = 1, 2, 3$, namely the subsequences $(x_{n_{\ell_{k_j}}}^i)$. We repeat the procedure until we obtain simultaneous subsequences of all m component sequences (x_n^i), which we can combine to form a subsequence of the original sequence (x_n) in $\mathbb{R}^m$.

3.13 Definition: Let $(a_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. We say that (a_n) is **Cauchy** when

$$\forall \epsilon > 0\ \exists N \in \mathbb{Z}_{\geq p}\ \forall k, \ell \in \mathbb{Z}_{\geq p} \Big(k, \ell \geq N \Longrightarrow |a_k - a_\ell| < \epsilon \Big).$$

3.14 Theorem: *(The Completeness of $\mathbb{R}^m$) For every sequence in $\mathbb{R}^m$, the sequence converges if and only if it is Cauchy.*

Proof: Let (x_n) be a sequence in $\mathbb{R}^m$. Suppose that (x_n) converges. Let $a = \lim\limits_{n\to\infty} x_n$. Let $\epsilon > 0$. Choose N so that $n \geq N \Longrightarrow |x_n - a| < \frac{\epsilon}{2}$. Then for $k, \ell \geq N$ we have $|x_k - a| < \frac{\epsilon}{2}$ and $|x_\ell - a| < \frac{\epsilon}{2}$ so $|x_k - x_\ell| \leq |x_k - a| + |a - x_\ell| < \epsilon$. Thus (x_n) is Cauchy.

Now suppose that $(x_n)_{n\geq p}$ is Cauchy. Choose $N \geq p$ so that $k, \ell \geq N \Longrightarrow |x_k - x_\ell| < 1$. Then for all $k \geq N$ we have $|x_k - x_N| < 1$ hence $|x_k| \leq |x_k - x_N| + |x_N| < 1 + |x_N|$, and so (x_n) is bounded by $\max\big\{|x_p|, |x_{p+1}|, \cdots, |x_{N-1}|, 1 + |x_N|\big\}$. Choose a convergent subsequence (x_{n_k}) and let $a = \lim\limits_{k\to\infty} x_{n_k}$. Let $\epsilon > 0$. Since (x_n) is Cauchy we can choose M so that $n, \ell \geq M \Longrightarrow |x_n - x_\ell| < \frac{\epsilon}{2}$. Since $\lim\limits_{k\to\infty} x_{n_k} = a$ we can choose k so that $n_k \geq M$ and $|x_{n_k} - a| < \frac{\epsilon}{2}$. Then for $n \geq M$ we have $|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \epsilon$.

3.15 Definition: Let $A \subseteq \mathbb{R}^\ell$ and let $f : A \to \mathbb{R}^m$. When a is a limit point of A and $b \in \mathbb{R}^m$, we say that $f(x)$ **converges to** b as x tends to a, and we write $\lim\limits_{x\to a} f(x) = b$ when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \Big(0 < |x - a| < \delta \Longrightarrow |f(x) - b| < \epsilon \Big).$$

When a is a limit point of A, we say that $f(x)$ **diverges to** ∞ and we write $\lim\limits_{x\to a} f(x) = \infty$ when

$$\forall r > 0\ \exists \delta > 0\ \forall x \in A \Big(0 < |x - a| < \delta \Longrightarrow |f(x)| \geq r \Big).$$

3.16 Theorem: *(Sequential Characterization of Limits) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let a be a limit point of A and let $u \in \mathbb{R}^m \cup \{\infty\}$. Then $\lim\limits_{x\to a} f(x) = u$ if and only if $\lim\limits_{n\to\infty} f(x_n) = u$ for every sequence (x_n) in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$.*

Proof: We give the proof in the case that $u \in \mathbb{R}^m$. Suppose first that $\lim\limits_{x\to a} f(x) = u \in \mathbb{R}^m$. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \to a$. Let $\epsilon > 0$. Since $\lim\limits_{x\to a} f(x) = u$ we can choose $\delta > 0$ so that $0 < |x - a| < \delta \Longrightarrow |f(x) - u| < \epsilon$. Since $x_n \to a$ we can choose N so that $n \geq N \Longrightarrow |x_n - a| < \delta$. For $n \geq N$ we have $|x_n - a| < \delta$ and we have $x_n \neq a$ (since $x_n \in A \setminus \{a\}$) and so $0 < |x_n - a| < \delta$ and hence $|f(x_n) - u| < \epsilon$. Thus $\lim\limits_{n\to\infty} f(x_n) = u$, as required.

Suppose, conversely, that $\lim\limits_{x\to a} f(x) \neq u$. Choose ϵ such that for every $\delta > 0$ there exists $x \in A$ such that $0 < |x - a| < \delta$ and $|f(x) - u| \geq \epsilon$. For each $n \in \mathbb{Z}^+$, choose $x_n \in A$ such that $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - u| \geq \epsilon$. For each n, since $0 < |x_n - a|$ we have $x_n \neq a$ so the sequence (x_n) lies in $A \setminus \{a\}$. Since $|x_n - a| < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ it follows that $x_n \to a$. Since $|f(x_n) - u| \geq \epsilon$ for all n, it follows that $\lim\limits_{n\to\infty} f(x_n) \neq u$. Thus we have found a sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ such that $\lim\limits_{n\to\infty} f(x_n) \neq u$.

3.17 Note: Using the Sequential Characterization of Limits, many properties of limits of sequences immediately imply analogous properties of limits of function. We list some of these properties in the following theorems.

3.18 Theorem: *(Uniqueness of Limits of Functions) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $a \in A'$, and let $u, v \in \mathbb{R}^m \cup \{\infty\}$. If $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} f(x) = v$ then $u = v$.*

Proof: This can be proven by imitating the proof of the Uniqueness of Limits of Sequences. Alternatively, we can use Uniqueness of Limits of Sequences together with the Sequential Characterization of Limits as follows. Since $a \in A'$ we can choose a sequence $(x_n) \in A \setminus \{a\}$ such that $x_n \to a$. By the Sequential Characterization of Limits, since $\lim\limits_{x\to a} f(x) = u$ we have $\lim\limits_{n\to\infty} f(x_n) = u$ and since $\lim\limits_{x\to a} f(x) = v$ we have $\lim\limits_{n\to\infty} f(x_n) = v$. By the Uniqueness of Limits of Sequences, since $\lim\limits_{n\to\infty} f(x_n) = u$ and $\lim\limits_{n\to\infty} f(x_n) = v$ it follows that $u = v$.

3.19 Theorem: *(Local Determination of Limits of Functions) Let $A \subseteq \mathbb{R}^\ell$, let $a \in A'$, let $B = B^*(a,r) \cap A$ with $r > 0$. Let $f : A \to \mathbb{R}^m$ and let $g : B \to \mathbb{R}^m$ and suppose that $f(x) = g(x)$ for all $x \in B$. Then $\lim\limits_{x\to a} f(x)$ exists in $\mathbb{R}^m \cup \{\infty\}$ if an only if $\lim\limits_{x\to a} g(x)$ exists in $\mathbb{R}^m \cup \{\infty\}$ and, in this case, the limits are equal.*

Proof: We leave the proof as an exercise.

3.20 Definition: Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$. We can write $f(x) = \big(f_1(x), f_2(x), \cdots, f_m(x)\big)$ where $f_k : A \to \mathbb{R}$ for each index k. Then the function f_k is called the k^{th} component function of f. Note that $f_k = p_k \circ f$ where $p_k : \mathbb{R}^m \to \mathbb{R}$ is the k **projection map** given by $p_k(y_1, \cdots, y_k, \cdots, y_m) = y_k$.

3.21 Theorem: *(Limits of Component Functions) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ be given by $f(x) = \big(f_1(x), \cdots, f_m(x)\big)$, let a be a limit point of A, and let $b = (b_1, b_2, \cdots, b_m) \in \mathbb{R}^m$. Then $\lim\limits_{x\to a} f(x) = b$ if and only if $\lim\limits_{x\to a} f_k(x) = b_k$ for all indices k.*

Proof: Suppose that $\lim\limits_{x\to a} f(x) = b$. Let (x_n) be any sequence in $A \setminus \{a\}$ with $x_n \to a$. By the Sequential Characterization of Limits, we have $\lim\limits_{n\to\infty} f(x_n) = b$. By Limits of Component Sequences, we have $\lim\limits_{n\to\infty} f_k(x_n) = b_k$ for all indices k. By the Sequential Characterization of Limits again, it follows that $\lim\limits_{x\to a} f_k(x) = b_k$ for all indices k.

Suppose, conversely, that $\lim\limits_{x\to a} f_k(x) = b_k$ for all k. Let (x_n) be any sequence in $A \setminus \{a\}$ with $x_n \to a$. By the Sequential Characterization of Limits, we have $\lim\limits_{n\to\infty} f_k(x_n) = b_k$ for all k. By Limits of Component Sequences, we have $\lim\limits_{n\to\infty} f(x) = b$. By the Sequential Characterization of Limits again, it follows that $\lim\limits_{x\to a} f(x) = b$.

3.22 Theorem: *(Operations on Limits of Functions) Let $f, g : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $a \in A'$ and let $c \in \mathbb{R}$. Suppose that $\lim\limits_{x\to a} f(x) = u \in \mathbb{R}^m$ and $\lim\limits_{n\to\infty} g(x) = v \in \mathbb{R}^m$. Then*

(1) $\lim\limits_{x\to a} (f+g)(x) = u + v$,
(2) $\lim\limits_{x\to a} (cf)(x) = cu$,
(3) $\lim\limits_{x\to a} |f|(x) = |u|$,
(4) $\lim\limits_{x\to a} (f \cdot g)(x) = u \cdot v$, *and*
(5) when $m = 3$ *we have* $\lim\limits_{x\to\infty} (f \times g)(x) = u \times v$.

Proof: This follows from Operations on Limits of Sequences, together with the Sequential Characterization of Limits.

3.23 Theorem: *(Comparison Theorem) Let $f, g : A \subseteq \mathbb{R}^\ell \to \mathbb{R}$ with $f(x) \leq g(x)$ for all $x \in A$ and let $a \in A'$.*

(1) If $\lim\limits_{x \to a} f(x) = u \in \mathbb{R} \cup \{\pm\infty\}$ and $\lim\limits_{x \to a} g(x) = v \in \mathbb{R} \cup \{\pm\infty\}$ then $u \leq v$.

(2) If $\lim\limits_{x \to a} f(x) = \infty$ then $\lim\limits_{x \to a} g(x) = \infty$.

(3) If $\lim\limits_{x \to a} g(x) = -\infty$ then $\lim\limits_{x \to \infty} f(x) = -\infty$.

Proof: This follows from the Comparison Theorem for Sequences in $\mathbb{R}$ together with the Sequential Characterization of Limits.

3.24 Theorem: *(Squeeze Theorem) Let $f, g, h : A \subseteq \mathbb{R}^\ell \to \mathbb{R}$ with $f(x) \leq g(x) \leq h(x)$ for all $x \in A$, and let $u \in \mathbb{R} \cup \{\pm\infty\}$. If $\lim\limits_{x \to a} f(x) = u = \lim\limits_{x \to a} h(x)$ then $\lim\limits_{x \to a} g(x) = u$.*

Proof: This follows from the Squeeze Theorem for Sequences in $\mathbb{R}$ together with the Sequential Characterization of Limits.

3.25 Definition: Let $A \subseteq \mathbb{R}^\ell$, let $B \subseteq \mathbb{R}^m$, and let $f : A \to B$. For $a \in A$, we say that f is **continuous at** a when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A\ \big(|x - a| < \delta \implies |f(x) - f(a)| < \epsilon\big).$$

We say that f is **continuous** (on A) when f is continuous at a for every $a \in A$. We say that f is **uniformly continuous** on A when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall a \in A\ \forall x \in A\ \big(|x - a| < \delta \implies |f(x) - f(a)| < \epsilon\big).$$

3.26 Theorem: *(Continuity at Limit Points and Isolated Points) Let $A \subseteq \mathbb{R}^\ell$ and let $f : A \to \mathbb{R}^m$.*

(1) When a is a limit point of A, f is continuous at $a \iff \lim\limits_{x \to a} f(x) = f(a)$.

(2) When a is an isolated point of A, f is always continuous at a.

Proof: We leave the proof as an exercise.

3.27 Theorem: *(Sequential Characterization of Continuity) Let $A \subseteq \mathbb{R}^\ell$, let $f : A \to \mathbb{R}^m$, and let $a \in A$. Then f is continuous at a if and only if $\lim\limits_{n \to \infty} f(x_n) = f(a)$ for every sequence $(x_n)_{n \geq p}$ in A with $\lim\limits_{n \to \infty} x_n = a$.*

Proof: Suppose f is continuous at a. Let (x_n) be any sequence in A with $x_n \to a$. Let $\epsilon > 0$. Since f is continuous at a we can choose $\delta > 0$ so that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. Since $x_n \to a$ we can choose N so that $n \geq N \implies |x_n - a| < \delta$. Then for all $n \geq N$ we have $|x_n - a| < \delta$ hence $|f(x_n) - f(a)| < \epsilon$, and so $\lim\limits_{n \to \infty} f(x_n) = f(a)$, as required.

Suppose that f is not continuous at a. Choose $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in A$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$. For each $n \in \mathbb{Z}^+$, choose $x_n \in A$ such that $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \geq \epsilon$. Since $|x_n - a| < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ it follows that $x_n \to a$. Since $\big|f(x_n) - f(a)\big| \geq \epsilon$ for all n, it follows that $\lim\limits_{n \to \infty} f(x_n) \neq f(a)$. Thus we have found a sequence (x_n) in A with $x_n \to a$ such that $\lim\limits_{x \to a} f(x_n) \neq f(a)$.

3.28 Theorem: *(Local Determination of Continuity) Let $A \subseteq \mathbb{R}^\ell$, let $a \in A'$, and let $B = B^*(a, r) \cap A$ where $r > 0$. Let $f : A \to \mathbb{R}^m$ and $g : B \to \mathbb{R}^m$ and suppose that $f(x) = g(x)$ for all $x \in B$. Then f is continuous at a if and only if g is continuous at a.*

Proof: The proof is left as an exercise.

3.29 Theorem: *(Continuity of Component Functions) Let $A \subseteq \mathbb{R}^\ell$ and let $f : A \to \mathbb{R}^m$. Then f is continuous at a if and only if f_k is continuous at a for every index k.*

Proof: This can be proven by imitating the proof of Continuity of Component Sequences or by using the result of Continuity of Component Sequences together with the Sequential Characterization of Continuity.

3.30 Theorem: *(Operations on Continuous Functions) Let $A \subseteq \mathbb{R}^\ell$, let $f, g : A \to \mathbb{R}^m$, let $a \in A$ and let $c \in \mathbb{R}$. If f and g are continuous at a then so are each of the functions $f+g$, cf, $|f|$ and $f \cdot g$, and also $f \times g$ in the case that $m = 3$.*

Proof: This follows from the Sequential Characterization of Continuity along with Operations on Limits of Sequences.

3.31 Theorem: *(Composition and Limits) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^p$ and let $h = g \circ f : C \subseteq \mathbb{R}^\ell \to \mathbb{R}^p$ where $C = A \cap f^{-1}(B)$. Let $a \in C' \subseteq A'$ and let $b \in B'$. Suppose that $\lim\limits_{x \to a} f(x) = b$ and $\lim\limits_{y \to b} g(y) = c \in \mathbb{R}^p \cup \{\infty\}$.*

(1) If $f(x) \neq b$ for all $x \in C \setminus \{a\}$ then $\lim\limits_{x \to a} h(x) = c$.

(2) If $b \in B$ and g is continuous at b then $\lim\limits_{x \to a} h(x) = g(b) = c$.

Proof: We leave the proof of Part (1) as an exercise. To prove Part (2), suppose that $b \in B$ and g is continuous at b. Note that since $b \in B'$ and g is continuous at b we have $g(b) = \lim\limits_{y \to b} g(y) = c$ by Theorem 3.26. Let (x_n) be any sequence in $C \setminus \{a\}$ with $x_n \to a$. Since $C \subseteq A$, the sequence (x_n) also lies in $A \setminus \{a\}$. By the Sequential Characterization of Limits of Functions, since $\lim\limits_{x \to a} f(x) = b$ we have $\lim\limits_{n \to \infty} f(x_n) = b$. For each index n we have $x_n \in C = A \cap f^{-1}(B)$ so that $f(x_n) \in B$, and so the sequence $\big(f(x_n)\big)$ lies in B. By the Sequential Characterization of Continuity, since g is continuous at b and $f(x_n) \to b$ we have $\lim\limits_{n \to \infty} g\big(f(x_n)\big) = g(b) = c$, that is $\lim\limits_{n \to \infty} h(x_n) = g(b) = c$. By the Sequential Characterization of Limits, it follows that $\lim\limits_{x \to a} h(x) = g(b) = c$.

3.32 Corollary: *(Composition of Continuous Functions) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^p$, and let $h = g \circ f : C \subseteq \mathbb{R}^\ell \to \mathbb{R}^p$ where $C = A \cap f^{-1}(B)$.*

(1) If f is continuous at $a \in A$ and g is continuous at $b = f(a) \in B$ then h is continuous at a.

(2) If f is continuous on A and g is continuous on B then h is continuous on C.

3.33 Definition: An **elementary function** is a function $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ which can be obtained, using the operations of addition, subtraction, multiplication, division, and composition of functions (whenever those operations are defined) from the following functions, which we call the **basic elementary functions**: the single-variable, real-valued functions c, x^n, $x^{1/n}$, e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$, and the k^{th} **inclusion map** $I_k : \mathbb{R} \to \mathbb{R}^\ell$ given by $I_k(t) = (0, \cdots, 0, t, 0, \cdots, 0) = t\, e_k$, and the k^{th} **projection map** $P_k : \mathbb{R}^\ell \to \mathbb{R}$ given by $P_k(x_1, \cdots, x_\ell) = x_k$.

3.34 Corollary: *Elementary functions are continuous in their domains.*

3.35 Exercise: Show that $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 - 2y^2}{x^2 + y^2}$, $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{x^2 + y^2}$ and $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy^2}{x^2 + y^4}$ do not exist, and that $\lim\limits_{(x,y) \to (0,0)} \dfrac{3x^2 y}{x^2 + 2y^2} = 0$ and $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}} = 0$.

3.36 Theorem: *(Topological Characterization of Continuity) Let $A \subseteq \mathbb{R}^n$, let $B \subseteq \mathbb{R}^m$, and let $f : A \to B$.*

(1) f is continuous if and only if $f^{-1}(E)$ is open in A for every open set E in B.
(2) f is continuous if and only if $f^{-1}(F)$ is closed in A for every closed set F in B.

Proof: We prove Part (1) and leave the proof of Part (2) as an exercise. Suppose that f is continuous. Let E be an open set in B. Let $a \in f^{-1}(E)$ so we have $f(a) \in E$. Since $f(a) \in E$ and E is open in B we can choose $\epsilon > 0$ so that $B_B\big(f(a), \epsilon\big) \subseteq E$. Since f is continuous at a we can choose $\delta > 0$ so that for all $x \in A$, $|x-a| < \delta \Longrightarrow \big|f(x) - f(a)\big| < \epsilon$. Let $x \in B_A(a, \delta)$, that is let $x \in A$ with $|x - a| < \delta$. Since $x \in A$ and $f : A \to B$ we have $f(x) \in B$. Since $x \in A$ with $|x - a| < \delta$, we have and $\big|f(x) - f(a)\big| < \epsilon$. Since $f(x) \in B$ with $\big|f(x) = f(a)\big| < \epsilon$, we have $f(x) \in B_B\big(f(a), \epsilon\big) \subseteq E$ hence $x \in f^{-1}(E)$. Since $a \in B_A(a, \delta)$ was arbitrary, this shows that $B_A(a, \delta) \subseteq f^{-1}(E)$. Thus $f^{-1}(E)$ is open in A, as required.

Suppose, on the other hand, that $f^{-1}(E)$ is open in A for every open set E in B. Let $a \in A$ and let $\epsilon > 0$. The set $E = B_B\big(f(a), \epsilon\big)$ is open in B so the set $f^{-1}(E)$ is open in A, and so we can choose $\delta > 0$ such that $B_A(a, \delta) \subseteq f^{-1}(E)$. It follows that for all $x \in B_A(a, \delta)$ we have $f(a) \in E = B_B\big(f(a), \epsilon\big)$. Equivalently, for all $x \in A$, if $|x - a| < \delta$ then $f(x) \in B$ with $\big|f(x) - f(a)\big| < \epsilon$. hus f is continuous at a. Since $a \in A$ was arbitrary, f is continuous (in its domain A).

3.37 Theorem: *(Properties of Continuous Functions) Let $\emptyset \neq A \subseteq \mathbb{R}^n$, let $B \subseteq \mathbb{R}^m$, and let $f : A \to B$ be continuous.*

(1) If A is connected then $f(A)$ is connected.
(2) If A is compact then $f(A)$ is compact.
(3) If A is compact then f is uniformly continuous on A.
(4) If A is compact and $m = 1$ then $f(x)$ attains its maximum and minimum values on A.
(5) if A is compact and f is bijective then f^{-1} is continuous.

Proof: We sketch a proof for Parts (1), (2) and (4) and leave some details, along with the other two parts, as an exercise. To prove Part (1), suppose that $f(A)$ is disconnected. Choose open sets U and V in $\mathbb{R}^m$ which separate $f(A)$. Since f is continuous and U and V are open, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are open in A. Verify that $f^{-1}(U)$ and $f^{-1}(V)$ separate A, so A is disconnected.

To prove Part (2), suppose that A is compact. Let $S = \big\{U_k \big| k \in K\big\}$ be an open cover of $f(A)$ (with each U_k open in $\mathbb{R}^n$). For each set $k \in K$, since U_k is open in $\mathbb{R}^m$ and f is continuous, it follows that $f^{-1}(U_k)$ is open in A. Let $T = \big\{f^{-1}(U_k) \big| k \in K\big\}$. Verify that T is an open cover of A (with each set $f^{-1}(U_k)$ open in A). Since A is compact, we can choose a finite subset $J \subseteq K$ such that the set $\big\{f^{-1}(U_j) \big| j \in J\big\}$ is an open cover of A. Verify that the set $\big\{U_j \big| j \in J\big\}$ is an open cover for $f(A)$, so $f(A)$ is compact.

To prove Part (4), suppose that $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A is compact. Since A is compact and f is continuous, $f(A)$ is compact by Part (2). Since $f(A)$ is compact, it is closed and bounded by the Heine Borel Theorem. Since $f(A)$ is bounded and non-empty (since $A \neq \emptyset$) it has a supremum and an infemum in $\mathbb{R}$. Let $u = \sup f(A)$. By the Approximation Property of the Supremum, for each $n \in \mathbb{Z}^+$ we can choose $x_n \in A$ with $u - \frac{1}{n} < f(x_n) \leq u$, and it follows that $f(x_n) \to u$ and hence u is a limit point of $f(A)$. Since u is a limit point of $f(A)$ and $f(A)$ is closed, we have $u \in f(A)$. Thus we can choose $a \in A$ such that $f(a) = u = \sup f(A) = \max f(A)$, and then f attains its maximum value at $a \in A$. Similarly, we can choose $b \in A$ such that $f(b) = \inf f(A) = \min f(A)$.

3.38 Definition: Let $A \subseteq \mathbb{R}^n$. For $a, b \in A$, the **line segment** between a and b is the set

$$[a,b] = \{a + t(b-a) \,|\, 0 \le t \le 1\}.$$

We say that A is **convex** when for every $a, b \in A$ we have $[a,b] \subseteq A$.

3.39 Example: Show that for $a \in \mathbb{R}^n$ and $r > 0$, the ball $B(a,r)$ is convex.

Proof: Let $b, c \in B(a,r)$ so we have $|b-a| < r$ and $|c-a| < r$. Let $x \in [b,c]$, say $x = b + t(c-b) = (1-t)b + tc$ with $0 \le t \le 1$. Note that

$$x - a = (1-t)b + tc - ((1-t)+t)a = (1-t)(b-a) + t(c-a).$$

By the Triangle Inequality, we have

$$\begin{aligned} |x-a| &= \big|(1-t)(b-a) + t(c-a)\big| \le \big|(1-t)(b-a)\big| + \big|t(c-a)\big| \\ &= (1-t)|b-a| + t|c-a| < (1-t)r + tr = r \end{aligned}$$

so that $x \in B(a,r)$. This shows that $[b,c] \subseteq B(a,r)$ and so $B(a,r)$ is convex.

3.40 Definition: Let $A \subseteq \mathbb{R}^n$ and let $a, b \in A$. A (continuous) **path** from a to b in A is a continuous function $f : [0,1] \to A$ with $f(0) = a$ and $f(1) = b$. We say that A is **path-connected** when for every $a, b \in A$ there exists a continuous path from a to b in A.

3.41 Note: For $A \subseteq \mathbb{R}^n$, if A is convex then A is path connected because given $a, b \in A$, since $[a,b] \subseteq A$, the map $f(t) = a + t(b-a)$ is a continuous path from a to b in A.

3.42 Theorem: *(Path-Connectedness and Connectedness) Let $A \subseteq \mathbb{R}^n$.*
(1) If A is path-connected then A is connected.
(2) If A is open and connected then A is path-connected.

Proof: We prove Part (1) and leave Part (2) as an exercise. Suppose that A is path connected and suppose, for a contradiction, that A is not connected. Let U and V be open sets in $\mathbb{R}^n$ which separate A, that is $U \cap A \ne \emptyset$, $V \cap A \ne \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$. Choose $a \in U \cap A$ and $b \in V \cap A$. Since A is path connected we can choose a continuous path $f : [0,1] \to A$ with $f(0) = a$ and $f(1) = b$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in $[0,1]$. Since $f(0) = a \in U$ we have $0 \in f^{-1}(U)$ so $f^{-1}(U) \ne \emptyset$. Similarly $1 \in f^{-1}(V)$ so $f^{-1}(V) \ne \emptyset$. Since $U \cap V = \emptyset$ we also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ (indeed if we had $t \in f^{-1}(U) \cap f^{-1}(V)$ then we would have $f(t) \in U$ and $f(t) \in V$ so that $f(t) \in U \cap V$). Since $f : [0,1] \to A \subseteq U \cup V$ it follows that $[0,1] = f^{-1}(U) \cup f^{-1}(V)$ (indeed, given $t \in [0,1]$ we have $f(t) \in A \subseteq U \cup V$, so either $f(t) \in U$ or $f(t) \in V$ hence either $t \in f^{-1}(U)$ or $t \in f^{-1}(V)$). Thus the open sets $f^{-1}(U)$ and $f^{-1}(V)$ separate $[0,1]$. This is not possible since $[0,1]$ is connected, so we have obtained the desired contradiction.

3.43 Example: Show that the set $U = \{(x,y) \in \mathbb{R}^2 \,|\, y > x^2\}$ is open in $\mathbb{R}^2$.

Solution: The map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = y - x^2$ is continuous (it is an elementary function), and the interval $I = (0, \infty)$ is open and so the set $U = f^{-1}(I)$ is open (by Theorem 3.36).

3.44 Example: Show that for $a \in \mathbb{R}^n$ and $r > 0$, the set $B(a,r)$ is connected.

Solution: Since $B(a,r)$ is convex (by Example 3.39), it is path connected (by Note 3.41), and hence it is connected (by Part 1 of Theorem 3.42).