Chapter 7. Introduction to Integrals

In this chapter we give an informal introduction to integration of functions of several variables. For functions of a single variable, the definite Riemann integral is defined for functions whose domain is a closed bounded interval. For functions of several variables, we are interested in functions with more varied domains. We do not allow our functions to have completely arbitrary domains, but we single out a restricted class of allowable domains, called **Jordan regions**, which have well-defined volumes. It is not immediately obvious that we need to place a restriction on the allowable domains, but it has been proven that there exist sets in $\mathbb{R}^n$ with rather unexpected properties which do not agree with our intuition about how a well-defined notion of volume ought to behave. Indeed, it was proven by Banach and Tarski that, in Euclidean space $\mathbb{R}^3$, there exist five sets $A_1, \cdots, A_5$ and five isometries $F_1, \cdots, F_5$, such that the sets A_k are disjoint and their union is a closed unit ball, but the congruent sets $F_k(A_k)$ are also disjoint, and their union is a union of two disjoint closed unit balls (so the volume of a disjoint union of sets need not be equal to the sum of the volumes of those sets).

7.1 Remark: We shall define **Jordan regions** and their **volumes** in the next chapter. For now, we state a theorem which provides some examples of Jordan regions.

7.2 Theorem: *In $\mathbb{R}$, every closed bounded interval*

$$[a,b] = \left\{x\in\mathbb{R} \,\middle|\, a\leq x\leq b\right\}$$

is a Jordan region. In $\mathbb{R}^2$, when $g,h:[a,b]\to\mathbb{R}$ are continuous with $g\leq h$ (that is with $g(x)\leq h(x)$ for all $x\in[a,b]$), the set

$$A = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a\leq x\leq b\,,\ g(x)\leq y\leq h(x)\right\}$$

is a Jordan region, and the set B obtained from A by interchanging x and y, that is

$$B = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a\leq y\leq b\,,\ g(y)\leq x\leq h(y)\right\}$$

is a Jordan region. In $\mathbb{R}^3$, when A is as above, and $k,\ell:A\to\mathbb{R}$ are continuous with $k\leq\ell$, the set

$$\begin{aligned} C &= \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, (x,y)\in A\,,\ k(x,y)\leq z\leq \ell(x,y)\right\} \\ &= \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, a\leq x\leq b\,,\ g(x)\leq y\leq h(x)\,,\ k(x,y)\leq z\leq \ell(x,y)\right\} \end{aligned}$$

is a Jordan region, and the various sets obtained from C by permuting the variables x, y and z are Jordan regions. When C is a closed Jordan region in $\mathbb{R}^{n-1}$ and $r,s:C\to\mathbb{R}$ are continuous with $r\leq s$, the set

$$D = \left\{(x,y)\in\mathbb{R}^n \,\middle|\, x\in C\,,\ r(x)\leq y\leq s(x)\right\}$$

is a Jordan region, and the sets obtained by permuting the variables $x_1,\cdots,x_{n-1},y$ are Jordan regions.

Also, the union of finitely many Jordan regions in $\mathbb{R}^n$ is a Jordan region on $\mathbb{R}^n$.

7.3 Remark: In the next chapter, we shall define what it means for a function $f:A\to\mathbb{R}$ to be (Riemann) **integrable**, where $A\subseteq\mathbb{R}^n$ is a Jordan region, and when f is integrable, we shall define its **integral** $\int_A f$. For now, we state some theorems which allow us to calculate the integrals of continuous functions on various Jordan regions, and to calculate the volumes of various Jordan regions.

7.4 Theorem: *(Integrability of Continuous Functions) When $A \subseteq \mathbb{R}^n$ is a closed Jordan region, every continuous function $f : A \to \mathbb{R}$ is integrable.*

7.5 Theorem: *(Iterated Integration) When $[a,b] = \left\{x\in\mathbb{R} \,\middle|\, a\le x\le b\right\}$ and $f : [a,b] \to \mathbb{R}$ is continuous, the integral of f on $[a,b]$ is the usual definite integral of f on $[a,b]$. that is*

$$\int_{[a,b]} f = \int_{x=a}^{b} f(x)\,dx.$$

When $A = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a\le x\le b\,,\ g(x)\le y\le h(x)\right\}$, where $g,h : [a,b] \to \mathbb{R}$ are continuous with $g \le h$, and when $f : A \to \mathbb{R}$ is continuous, the integral of f on A is given by

$$\int_A f = \int_{x=a}^{b}\left(\int_{y=g(x)}^{h(x)} f(x,y)\,dy\right)dx.$$

Similarly, when $B = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a\le y\le b\,,\ g(y)\le x\le h(y)\right\}$, where $g,h : [a,b] \to \mathbb{R}$ are continuous, and when $f : B \to \mathbb{R}$ is continuous, we have

$$\int_B f = \int_{y=a}^{b}\left(\int_{x=g(y)}^{h(y)} f(x,y)\,dx\right)dy.$$

When $C = \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, a\le x\le b\,,\ g(x)\le y\le h(x)\,,\ k(x,y)\le z\le \ell(x,y)\right\}$ where $g,h : [a,b] \to \mathbb{R}$ and $k,\ell : A \to \mathbb{R}$ are continuous with $g \le h$ and $k \le \ell$, and when $f : C \to \mathbb{R}$ is continuous, the integral of f on C is given by

$$\int_C f = \int_{x=a}^{b}\left(\int_{y=g(x)}^{h(x)}\left(\int_{z=k(x,y)}^{\ell(x,y)} f(x,y,z)\,dz\right)dy\right)dx,$$

and similar formulas hold in the case that the variables x, y and z are permuted.
When $D = \left\{(x,y)\in\mathbb{R}^n \,\middle|\, x\in C\,,\ r(x)\le y\le s(x)\right\}$ where $C \subseteq \mathbb{R}^{n-1}$ is a closed Jordan region in $\mathbb{R}^{n-1}$ and $r,s : C \to \mathbb{R}$ are continuous with $r \le s$, and when $f : D \to \mathbb{R}$ is continuous, we have

$$\int_D f = \int_C\left(\int_{y=r(x)}^{s(x)} f(x,y)\,dy\right)dx,$$

and similar formulas hold in the case that the variables $x_1, x_2, \cdots, x_{n-1}, y$ are permuted.

7.6 Theorem: *(Decomposition) Let $A = \bigcup_{k=1}^m A_k$, where $A_1, A_2, \cdots, A_m \subseteq \mathbb{R}^n$ are closed Jordan regions in $\mathbb{R}^n$ with disjoint interiors, and let $f : A \to \mathbb{R}$ be continuous. Then*

$$\int_A f = \sum_{k=1}^{m}\int_{A_k} f\,.$$

7.7 Theorem: *(Integration and Volume) Let $A \subseteq \mathbb{R}^n$ be a Jordan region. Then the (n-dimensional)* **volume** *of A is given by*

$$\mathrm{Vol}(A) = \int_A 1\,.$$

7.8 Definition: *When A is a Jordan region in $\mathbb{R}$, the (1-dimensional) volume of A is also called the* **length** *of A, and when A is a Jordan region in $\mathbb{R}^2$, the (2-dimensional) volume of A is also called the* **area** *of A.*

7.9 Note: *When $D = [a, b] \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous, the integral $\int_D f$ measures the* **signed area** *of the region in $\mathbb{R}^2$ between the graph of f and the x-axis. In the case that $D = [a, b]$ represents the position of a long thin solid object (like a string or a wire) and the function f represents its* **linear density** *(mass per unit length), or its* **linear charge density** *(charge per unit length), the integral $\int_D f$ measures the total* **mass**, *or the total* **charge** *of the object.*

When $D \subseteq \mathbb{R}^2$ is a Jordan region and $f : D \to \mathbb{R}$, the integral $\int_D f$ measures the **signed volume** *of the region between the graph of f and the xy-plane. In the case that $D \subseteq \mathbb{R}^2$ represents the shape of a flat object and the function $f : D \to \mathbb{R}$ represents its* **planar density** *(mass per unit area), or its* **planar charge density** *(charge per unit area), the integral $\int_D f$ measures the total* **mass**, *or the total* **charge**, *of the object.*

When $D \subseteq \mathbb{R}^3$ is a Jordan region representing the shape of a solid object, and $f : D \to \mathbb{R}$ represents its **density** *(mass per unit volume), or its* **charge density** *(charge per unit volume), the integral $\int_D f$ measures the total* **mass**, *or the total* **charge**, *of the object.*

7.10 Notation: *Integrals can be denoted in a number of different ways. For example, when $D = [a, b] \subseteq \mathbb{R}$ we can write*

$$\int_D f = \int_D f\, dL = \int_D f(x)\, dL = \int_D f(x)\, dx = \int_{x=a}^{b} f(x)\, dx,$$

(where the letter L stands for length), when $D \subseteq \mathbb{R}^2$ we can write

$$\int_D f = \int_D f\, dA = \int_D f(x, y)\, dA = \int_D f(x, y)\, dx\, dy$$

(where A stands for area), or we can use two integral signs and write

$$\int_D f = \iint_D f = \iint_D f\, dA = \iint_D f(x, y)\, dA = \iint_D f(x, y)\, dx\, dy,$$

when $D \subseteq \mathbb{R}^3$ we can write

$$\int_D f = \int_D f\, dV = \int_D f(x, y, z)\, dV = \int_D f(x, y, z)\, dx\, dy\, dz$$

(where V stands for volume), or we can use three integral signs and write

$$\int_D f = \iiint_D f = \iiint_D f\, dV = \iiint_D f(x, y, z)\, dV = \iiint_D f(x, y, z)\, dx\, dy\, dz,$$

and when $D \subseteq \mathbb{R}^n$ we can write

$$\int_D f = \int_D f\, dV = \int_D f(x)\, dV = \int_D f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2\, \cdots\, dx_n$$

or we can use multiple integral signs.

7.11 Exercise: *Let T be the triangle in $\mathbb{R}^2$ with vertices at $(0, -1)$, $(2, 1)$ and $(2, 3)$. Find $\int_T 2xy\, dA$.*

7.12 Exercise: *Find the volume of the region in $\mathbb{R}^3$ which lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2x$.*

7.13 Exercise: *Find the mass of the tetrahedron with vertices at $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$ and $(2, 2, 2)$ given that the density is given by $\rho(x, y, z) = 2xy(3 - z)$.*

7.14 Definition: *Let $C, D \subseteq \mathbb{R}^n$ be closed Jordan regions. An* **orientation preserving change of coordinates map** *from C to D is a continuous map $g : C \to D$ such that the map $g : C^o \to D^o$ is invertible and $\mathcal{C}^1$ with $\det\big(Dg(x)\big) > 0$ for all $x \in C^o$, and an* **orientation reversing change of coordinates map** *from C to D is a continuous map $g : C \to D$ such that $g : C^o \to D^o$ is invertible and $\mathcal{C}^1$ with $\det\big(Dg(x)\big) < 0$ for all $x \in C^o$.*

7.15 Example: *Three important orientation preserving change of coordinates maps are the* **polar coordinates map** *in $\mathbb{R}^2$, which is given by*

$$(x, y) = g(r, \theta) = \big(r\cos\theta\,,\, r\sin\theta\big) \text{ with } \det Dg(r, \theta) = r\,,$$

the **cylindrical coordinates map** *in $\mathbb{R}^3$, which is given by*

$$(x, y, z) = g(r, \theta, z) = \big(r\cos\theta\,,\, r\sin\theta\,,\, z\big) \text{ with } \det Dg(r, \theta, z) = r,$$

and the **spherical coordinates map** *in $\mathbb{R}^3$, which is given by*

$$(x, y, z) = g(r, \phi, \theta) = \big(r\,\sin\phi\,\cos\theta\,,\, r\,\sin\phi\,\sin\theta\,,\, r\,\cos\phi\big) \text{ with } \det Dg(r, \phi, \theta) = r^2\sin\phi.$$

7.16 Theorem: *(Change of Variables) Let $C, D \subseteq \mathbb{R}^n$ be closed Jordan regions and let $g : C \to D$ be a change of coordinates map from C to D. Then*

$$\int_D f = \int_C (f \circ g)\big|\det Dg\big|.$$

Proof: We may state and prove a slightly different version of this theorem in Chapter 9.

7.17 Example: When $D = [a, b] \subseteq \mathbb{R}$, and $g : C \subseteq \mathbb{R} \to D \subseteq \mathbb{R}$ is a change of variables map from C to D given by $x = g(u)$ with inverse $u = h(x)$, and $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous, we have

$$\int_{x=a}^{b} f(x)\,dx = \int_D f(x)dx = \int_C f\big(g(u)\big)\big|\det Dg(u)\big|\;du = \int_{u=h(a)}^{h(b)} f\big(g(u)\big)g'(u)\;du.$$

When $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous and $g : C \subseteq \mathbb{R}^2 \to D \subseteq \mathbb{R}^2$ is a change of variables map from C to D given by $(x, y) = g(u, v)$, we have

$$\iint_D f(x, y)\,dx\,dx = \iint_C f\big(g(u, v)\big)\big|\det Dg(u, v)\big|\;du\,dv.$$

When $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is continuous and $g : C \subseteq \mathbb{R}^3 \to D \subseteq \mathbb{R}^3$ is a change of variables map from C to D given by $(x, y, z) = g(u, v, w)$, we have

$$\iiint_D f(x, y, z)\;dx\,dy\,dz = \iiint_C f\big(g(u, v, w)\big)\big|\det Dg(u, v, w)\big|\;du\,dv\,dw.$$

7.18 Exercise: Find the area inside the cardioid $r = 2 + 2\cos\theta$.

7.19 Exercise: Find the volume of the region which lies inside the sphere $x^2 + y^2 + z^2 = 4$ and inside the cylinder $x^2 - 2x + y^2 = 0$.

7.20 Exercise: Find the mass of the ball $x^2 + y^2 + z^2 \leq 4$ given that the density is given by $\rho(x, y, z) = 1 - \frac{1}{2}\sqrt{x^2 + y^2 + z^2}$.

7.21 Exercise: Find the volume of the region under the graph of $z = e^{-(x^2+y^2)}$.

7.22 Definition: Let $n = 2$ or 3, let $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be continuous on $[a,b]$ and $\mathcal{C}^1$ in (a,b), let C be the curve in $\mathbb{R}^n$ which is given parametrically by $(x,y) = \alpha(t)$ or by $(x,y,z) = \alpha(t)$ for $a \le t \le b$, and let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous on $C = \text{Range}(\alpha)$. Then we write $dL = \big|\alpha'(t)\big|\,dt$ and we define the (curve) **integral of f on C** to be

$$\int_\alpha f\ dL = \int_C f\ dL = \int_{t=a}^b f\big(\alpha(t)\big)\big|\alpha'(t)\big|\,dt.$$

When C is a union $C = \bigcup\limits_{k=1}^m C_k$ of curves C_k as above, we define $\displaystyle\int_C f dL = \sum_{k=1}^m \int_{C_k} f\,dL$.

Let D be a closed Jordan region in $\mathbb{R}^2$, let $\sigma : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be continuous in D and $\mathcal{C}^1$ in D^o, let S be the surface in $\mathbb{R}^3$ which is given parametrically by $(x,y,z) = \sigma(s,t)$, and let $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$ be continuous on S. Then we write $dA = \big|\sigma_s \times \sigma_t\big|\,ds\,dt$ and we define the (surface) **integral of f on S** to be

$$\iint_\sigma f\ dA = \iint_S f\ dA = \iint_D f\big(\sigma(s,t)\big)\big|\sigma_s(s,t) \times \sigma_t(s,t)\big|\,ds\,dt.$$

where $\sigma_s(s,t) = \left(\frac{\partial x}{\partial s}(s,t), \frac{\partial y}{\partial s}(s,t), \frac{\partial z}{\partial s}(s,t)\right)^T$ and $\sigma_t(s,t) = \left(\frac{\partial x}{\partial s}(s,t), \frac{\partial y}{\partial s}(s,t), \frac{\partial z}{\partial s}(s,t)\right)^T$.

When S is a union $S = \bigcup\limits_{k=1}^m S_k$ of surfaces S_k as above, we define $\displaystyle\int_S f dA = \sum_{k=1}^m \int_{S_k} f\ dA$.

7.23 Note: When C is a curve in $\mathbb{R}^n$ with $n = 2$ or 3, which is given by $(x,y) = \alpha(t)$ or by $(x,y,z) = \alpha(t)$ for $a \le t \le b$, the integral of the constant function 1 on C measures the **length** (or **arclength**) of the curve C, and in the case that C represents the shape of a physical object and the function $f : C \subseteq \mathbb{R}^2 \to \mathbb{R}$ represents its linear density (or linear charge density), the integral of f on C measures the total **mass** (or **charge**) of the object.

When S is a surface in $\mathbb{R}^3$ which is given by $(x,y,z) = \sigma(s,t)$ for $(s,t) \in D \subseteq \mathbb{R}^2$, the integral of the constant function 1 on S measures the **area** (or **surface-area**) of the surface S, and in the case that S represents the shape of a physical object and the function $f : S \to \mathbb{R}$ represents its surface **density** (or surface **charge density**), the integral of f on S measures the total **mass** (or **charge**) of the surface.

7.24 Exercise: Find the arclength of the helix $\alpha(t) = \big(t, \cos t, \sin t\big)$ for $0 \le t \le 2\pi$.

7.25 Exercise: Find the surface area of the torus given by

$$(x,y,z) = \sigma(\theta,\phi) = \Big((2+\cos\phi)\cos\theta\,, (2+\cos\phi)\sin\theta\,,\ \sin\phi\Big)$$

for $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$.

7.26 Exercise: Find the mass of the hollow sphere $x^2 + y^2 + z^2 = 1$ when the surface density (mass per unit area) is given by $\rho(x,y,z) = 3 - z$.

7.27 Exercise: Find the mass of the curve of intersection of the parabolic sheet $z = x^2$ with the paraboloid $z = 2 - x^2 - 2y^2$, when the linear density (mass per unit length) is given by $\rho(x,y,z) = |xy|$.