MATH 247 Calculus 3, Solutions to the Exercises for Chapter 2

1: Let $0 \neq u, v, w \in \mathbf{R}^n$.

(a) (Trigonometric Ratios) Show that if $(v-u)\cdot u = 0$ then $\cos\theta(u,v) = \frac{|u|}{|v|}$ and $\sin\theta(u,v) = \frac{|v-u|}{|v|}$

Solution: Suppose that $(v-u)\cdot u = 0$ and let $\theta = \theta(u,v)$. We have $0 = (v-u)\cdot u = v\cdot u - u\cdot u = v\cdot u - |u|^2$ so that $u\cdot v = |u|^2$ and hence

$$\cos\theta = \frac{u\cdot v}{|u|\,|v|} = \frac{|u|^2}{|u|\,|v|} = \frac{|u|}{|v|}\,.$$

Also, we have $|v-u|^2 = (v-u)\cdot(v-u) = |v|^2 - 2(u\cdot v) + |u|^2 = |v|^2 - 2|u|^2 + |u|^2 = |v|^2 - |u|^2$ and so

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{|u|^2}{|v|^2} = \frac{|v|^2 - |u|^2}{|v|^2} = \frac{|v-u|^2}{|v|^2}\,.$$

Since $\theta \in [0,\pi]$ we have $\sin\theta \geq 0$, and so taking the square root on both sides gives

$$\sin\theta = \frac{|v-u|}{|v|}\,.$$

(b) (Angle Addition) Show that if $0 \neq w = su + tv$ for some $s, t \geq 0$ then we have $\theta(u,v) = \theta(u,w) + \theta(w,v)$.

Solution: First we note that when $t > 0$ we have

$$\theta(tu,v) = \cos^{-1}\frac{(tu)\cdot v}{|tu|\,|v|} = \cos^{-1}\frac{t(u\cdot v)}{t|u|\,|v|} = \cos^{-1}\frac{u\cdot v}{|u|\,|v|} = \theta(u,v)$$

and similarly $\theta(u,tv) = \theta(u,v)$. It follows that for $\hat{u} = \frac{u}{|u|}$ and $\hat{v} = \frac{v}{|v|}$ we have $\theta(u,v) = \theta(\hat{u},\hat{v})$. Also note that if $0 \neq w = su + tv$ with $s,t \geq 0$ then for $\hat{w} = \frac{w}{|w|}$ we have $\hat{w} = \frac{s|u|}{|w|}\,\hat{u} + \frac{t|v|}{|w|}\,\hat{v}$. It follows that it suffices to consider the case that u, v and w are unit vectors (since, if necessary, we can replace them by $\hat{u}$, $\hat{v}$ and $\hat{w}$).

Suppose that u, v and w are unit vectors. Then

$$1 = |w|^2 = (su+tv)\cdot(su+tv) = s^2 + 2st(u\cdot v) + t^2.$$

We have

$$\begin{aligned}
\cos\theta(u,w) &= u\cdot w = u\cdot(su+tv) = s + t(u\cdot v)\\
\sin\theta(u,w) &= \sqrt{1-\cos^2\theta(u,w)} = \sqrt{1-(s+t(u\cdot v))^2}\\
&= \sqrt{(s^2+2st(u\cdot v)+t^2) - (s^2+2st(u\cdot v)+t^2(u\cdot v)^2}\\
&= \sqrt{t^2 - t^2(u\cdot v)^2} = t\sqrt{1-(u\cdot v)^2}
\end{aligned}$$

and similarly

$$\begin{aligned}
\cos\theta(v,w) &= t + s(u\cdot v)\\
\sin\theta(v,w) &= s\sqrt{1-(u\cdot v)^2}
\end{aligned}$$

and so

$$\begin{aligned}
\cos\big(\theta(u,w)+\theta(v,w)\big) &= (s+t(u\cdot v))(t+s(u\cdot v)) - t\sqrt{1-(u\cdot v)^2}\cdot s\sqrt{1-(u\cdot v)^2}\\
&= \big(st + s^2(u\cdot v) + t^2(u\cdot v) + st(u\cdot v)^2\big) - st\big(1-(u\cdot v)^2\big)\\
&= s^2(u\cdot v) + t^2(u\cdot v) + 2st(u\cdot v)^2\\
&= \big(s^2+t^2+2st(u\cdot v)\big)(u\cdot v)\\
&= u\cdot v = \cos\theta(u,v)\ \text{, and}\\
\sin\big(\theta(u,w)+\theta(v,w)\big) &= \sin\theta(u,w)\cos\theta(v,w) + \cos\theta(u,w)\sin\theta(v,w)\\
&= t\sqrt{1-(u\cdot v)^2}\big(t+s(u\cdot v)\big) + \big(s+t(u\cdot v)\big)\cdot s\sqrt{1-(u\cdot v)^2}\\
&= \big(t^2 + st(u\cdot v) + s^2 + st(u\cdot v)\big)\sqrt{1-(u\cdot v)^2}\\
&= \sqrt{1-(u\cdot v)^2} = \sin\theta(u,v).
\end{aligned}$$

2: (a) Let $A=\{(x,y)\in\mathbf{R}^2\,\big|\,0<x,\ 0<y \text{ and } xy<1\}$. Show, from the definition of an open set, that A is open in $\mathbf{R}^2$.

Solution: Before beginning our proof, let us discuss our strategy. Suppose that $(a,b)\in A$, so we have $a>0$, $b>0$ and $ab<1$. We want to choose $r>0$ so that the disc $B_r=B\big((a,b),r\big)$ is contained in A. Note that the open square Q_r given by $|x-a|<r$ and $|y-b|<r$ contains the disc B_r, so it suffices to ensure that Q_r is contained in A. Note that if $r<a$ then $|x-a|<r\implies |x-a|<a\implies 0<x<2a\implies x>0$. Similarly, if $r<b$ then $|y-b|<r\implies y>0$. Note that if $r<a$ and $r<b$ then $r<a+b$ and so $(a+r)(b+r)=ab+r(a+b)+r^2<ab+r(a+b)+r(a+b)=ab+2r(a+b)$ and we can obtain $(a+r)(b+r)<1$ by choosing $r<\frac{1-ab}{2(a+b)}$.

Now we begin the proof. Let $(a,b)\in A$, so we have $a>0$, $b>0$ and $ab<1$. Choose $r=\min\left\{a,b,\frac{1-ab}{2(a+b)}\right\}$. Let $(x,y)\in B_r=B\big((a,b),r\big)$. Then $|x-a|=\sqrt{|x-a|^2}\leq\sqrt{|x-a|^2+|y-b|^2}=\big|(x,y)-(a,b)\big|<r$ and similarly $|y-b|<r$. Since $|x-a|<r\leq a$ we have $0\leq a-r<x<a+r$ and since $|y-b|<r\leq b$ we have $0\leq b-r<y<b+r$. Since $0<x<a+r$ and $0<y<a+r$ and $r<a+b$ and $r<\frac{1-ab}{2(a+b)}$ we have $xy<(a+r)(b+r)=ab+r(a+b)+r^2<ab+2r(a+b)<ab+(1-ab)=1$. Since $x>0$ and $y>0$ and $xy<1$ we have $(x,y)\in A$. Thus $B_r\subseteq A$, as required, and so A is open.

(b) Let $B=\left\{\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right)\in\mathbf{R}^2\,\middle|\,t\in\mathbf{R}\right\}$. Show that B is not closed in $\mathbf{R}^2$.

Solution: To solve this problem, you might find it helpful to draw a picture of the set B by choosing various values of t and plotting points. You should find that B looks like the unit circle centred at $(0,0)$ with the point $(0,1)$ removed. If you wish, you can show, algebraically, that this is indeed the case.

Let $a=(0,1)$. Let $x(t)=\frac{2t}{t^2+1}$ and $y(t)=\frac{t^2-1}{t^2+1}$ and $f(t)=\big(x(t),y(t)\big)$ so that $B=\big\{f(t)\big|t\in\mathbf{R}\big\}$. We claim that $a\in B'$ (that is a is a limit point of B) but $a\notin B$. It is clear that $a\notin B$ because to get $f(t)=a$ we need $x(t)=0$ and $y(t)=1$, but to get $x(t)=\frac{2t}{t^2+1}=0$ we must choose $t=0$, and then $y(t)=\frac{t^2-1}{t^2+1}=-1\neq 1$. To show that $a\in B'$, we shall show that for all $r>0$ we have $B(a,r)\cap B\neq\emptyset$. Let $r>0$. Since $\lim\limits_{t\to\infty}x(t)=0$ and $\lim\limits_{t\to\infty}y(t)=1$ we can choose $t\in\mathbf{R}$ so that $\big|x(t)-0\big|<\frac{r}{2}$ and $\big|y(t)-1\big|<\frac{r}{2}$. Then we have

$$\big|f(t)-a\big|=\big|(x(t),y(t))-(0,1)\big|=\big|\big(x(t)\,,\,y(t)-1\big)\big|\leq|x(t)|+|y(t)-1|<\tfrac{r}{2}+\tfrac{r}{2}=r$$

and so $f(t)\in B(a,r)\cap B$. This shows that for all $r>0$ we have $B(a,r)\cap B\neq\emptyset$, and so $a\in B'$. Since $a\in B'$ and $a\notin B$ we do not have $B'\subseteq B$ and so B is not closed (by Part (2) of Theorem 2.19).

3: Let $A\subseteq\mathbf{R}^n$.

(a) Show that A' is closed in $\mathbf{R}^n$.

Solution: By Part (2) of Theorem 2.19, we know that A' is closed if and only if $(A')'\subseteq A'$. Let $a\in(A')'$, that is let a be a limit point of A'. Let $r>0$. Since a is a limit point of A', we know that $B^*(a,r)\cap A'\neq\emptyset$. Choose $b\in B^*(a,r)\cap A'$. Note that $0<|a-b|<r$. Let $s=\min\big(|a-b|\,,r-|a-b|\big)>0$. Since $b\in A'$ we know that $B^*(b,s)\cap A\neq\emptyset$. Choose $c\in B^*(b,s)\cap A$. We claim that $c\in B^*(a,r)\cap A$. By the Triangle Inequality we have $|a-c|\leq|a-b|+|b-c|<|a-b|+s\leq|a-b|+r-|a-b|=r$, and by the Triangle Inequality again, we have $|a-b|\leq|a-c|+|c-b|$ and so $|a-c|\geq|a-b|-|b-c|>|a-b|-s\geq|a-b|-|a-b|=0$. Thus $0<|a-c|<r$ and so $c\in B^*(a,r)\cap A$, as claimed. Since $c\in B^*(a,r)\cap A$, we see that $B^*(a,r)\cap A\neq\emptyset$. We have shown that for every $r>0$ we have $B^*(a,r)\cap A\neq\emptyset$, and so $a\in A'$. This proves that $(A')'\subseteq A'$, and so A' is closed.

(b) Show that $\partial A=\overline{A}\setminus A^o$.

Solution: Let $a\in\partial A$. We claim first that $a\in\overline{A}$. Since $\overline{A}=A\cup A'$ it suffices to show that either $a\in A$ or $a\in A'$. Suppose that $a\notin A$. Let $r>0$ be arbitrary. Since $a\in\partial A$ we have $B(a,r)\cap A\neq\emptyset$. Since $a\notin A$ we have $B^*(a,r)\cap A=B(a,r)\cap A$ and so $B^*(a,r)\cap A)\neq\emptyset$. Since $r>0$ was arbitrary, we have $a\in A'$, as required.

Next we claim that $a\notin A^0$. Suppose, for a contradiction, that $a\in A^0$. By Part (b), a is an interior point of A so we can choose $r>0$ so that $B(a,r)\subseteq A$. Since $B(a,r)\subseteq A$ we have $B(a,r)\cap A^c=\emptyset$. But since $a\in\partial A$ we have $B(a,r)\cap A^c\neq\emptyset$, so we have obtained the desired contradiction. Thus $a\notin A^0$, as claimed. This completes the proof that $\partial A\subseteq\overline{A}\setminus A^0$.

Now let $a\in\overline{A}\setminus A^0$, that is let $a\in\overline{A}$ with $a\notin A^0$. Let $r>0$ be arbitrary. Case 1: suppose that $a\in A$. Let $r>0$ be arbitrary. Since $a\in A$ and $a\in B(a,r)$ we have $B(a,r)\cap A\neq\emptyset$. Since $a\notin A^0$ we have $B(a,r)\not\subseteq A$ and so $B(a,r)\cap A^c\neq\emptyset$. Thus $a\in\partial A$. Case 2: suppose that $a\notin A$. Let $r>0$ be arbitrary. Since $a\notin A$ and $a\in B(a,r)$ we have $B(a,r)\cap A^c\neq\emptyset$. Since $a\in\overline{A}=A\cup A'$ and $a\notin A$ we have $a\in A'$ and so $B^*(a,r)\cap A\neq\emptyset$ hence $B(a,r)\cap A\neq\emptyset$. Thus $a\in\partial A$. In either case we find that $a\in\partial A$. This completes the proof that $\overline{A}\setminus A^0\subseteq\partial A$.

4: (a) Let $A, B \subseteq \mathbf{R}^n$ show that if A is connected and $A \subseteq B \subseteq \overline{A}$ then B is connected.

Solution: Suppose that A is connected and that $A \subseteq B \subseteq \overline{A}$. Suppose, for a contradiction, that B is disconnected. Choose open sets $U, V \subseteq \mathbf{R}^n$ which separate B, so we have $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $U \cap B = \emptyset$ and $B \subseteq U \cup V$. We claim that U and V also separate A (contradicting the fact that A is connected). Since $A \subseteq B \subseteq U \cup V$, it suffices to prove that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$. We claim that $U \cap A \neq \emptyset$. Since $U \cap B \neq \emptyset$ we can choose $b \in U \cap B$. Then we have $b \in B \subseteq \overline{A} = A \cup A'$, and so either $b \in A$ or $b \in A'$. If $b \in A$ then we have $b \in U \cap A$ so that $U \cap A \neq \emptyset$. Suppose that $b \in A'$. Since $b \in U$ and U is open, we can choose $r > 0$ such that $B(b, r) \subseteq U$. Since $b \in A'$ we have $B(b, r) \cap A \neq \emptyset$ so we can choose $c \in B(b, r) \cap A$. Then we have $c \in B(b, r) \subseteq U$ and $c \in A$, hence $c \in U \cap A$, and so $U \cap A \neq \emptyset$. This proves that $U \cap A \neq \emptyset$, as claimed. The proof that $V \cap A \neq \emptyset$ is similar, and so U and V separate A giving the desired contradiction.

(b) Let S be a nonempty set and let $A_j \subseteq \mathbf{R}^n$ for each $j \in S$. Suppose that A_j is connected for all $j \in S$ and that $A_k \cap A_\ell \neq \emptyset$ for all $k, \ell \in S$. Show that $\bigcup_{j \in S} A_j$ is connected.

Solution: Let $B = \bigcup_{j \in S} A_j$. Suppose, for a contradiction, that B is disconnected. Choose open sets $U, V \subseteq \mathbf{R}^n$ which separate B, that is $B \cap U \neq \emptyset$, $B \cap V \neq \emptyset$, $U \cap V = \emptyset$ and $B \subseteq U \cup V$. Choose $a \in B \cap U$ and $b \in B \cap V$. Since $a \in B = \bigcup_{j \in S} A_j$, we can choose $k \in S$ such that $a \in A_k$. Similarly we can choose $\ell \in S$ such that $b \in A_\ell$. Then we have $a \in A_k \cap U$ and $b \in A_\ell \cap V$. Since A_k is connected, and $a \in A_k \cap U$ so that $A_k \cap U \neq \emptyset$, and $A_k \subseteq \bigcup_{j \in S} A_j = B \subseteq U \cup V$, it follows that we must have $A_k \subseteq U$ because otherwise we would have $A_k \cap V \neq \emptyset$ and so U and V would separate A_k. Similarly, we must have $A_\ell \subseteq V$. Since $A_k \subseteq U$ and $A_\ell \subseteq V$ we have $A_k \cap A_\ell \subseteq U \cap V = \emptyset$. This contradicts our assumption that $A_k \cap A_\ell \neq \emptyset$, and so B is connected, as required.

5: Let $A \subseteq P \subseteq \mathbf{R}^n$. Define the **interior of** A **in** P to be the union of all sets $E \subseteq P$ such that E is open in P and $E \subseteq A$. Define the **closure of** A **in** P to be the intersection of all sets $F \subseteq P$ such that F is closed in P and $A \subseteq F$. Denote the interior of A in $\mathbf{R}^n$ and the closure of A in $\mathbf{R}^n$ by A^o and $\overline{A}$ (as usual). Denote the interior of A in P and the closure of A in P by $\mathrm{Int}_P(A)$ and $\mathrm{Cl}_P(A)$.

(a) Show that $\mathrm{Cl}_P(A) = \overline{A} \cap P$.

Solution: Since $\overline{A}$ is closed in $\mathbf{R}^n$ it follows that $\overline{A} \cap P$ is closed in P. Since $A \subseteq \overline{A}$ and $A \subseteq P$ we have $A \subseteq \overline{A} \cap P$. Since $\overline{A} \cap P$ is closed in P and $A \subseteq \overline{A} \cap P$, it follows from the definition of $\mathrm{Cl}_P(A)$ that $\mathrm{Cl}_P(A) \subseteq \overline{A} \cap P$.

Let F be any closed set in P with $A \subseteq F$. Choose a closed set K in $\mathbf{R}^n$ such that $F = K \cap P$. Since K is closed in $\mathbf{R}^n$ and $A \subseteq K$ we have $\overline{A} \subseteq K$. Thus $\overline{A} \cap P \subseteq K \cap P = F$. Since $\overline{A} \cap P \subseteq F$ for every closed set F in P which contains A, it follows, from the definition of $\mathrm{Cl}_P(A)$, that $\overline{A} \cap P \subseteq \mathrm{Cl}_P(A)$.

(b) Show that $\mathrm{Int}_P(A) = (A \cup P^c)^o \cap P$, where $P^c = \mathbf{R}^n \setminus P$.

Solution: Let $F = (A \cup P^c)^o \cap P$. Since $(A \cup P^c)^o$ is open in $\mathbf{R}^n$ it follows that $F = (A \cup P^c)^o \cap P$ is open in P. Also note that we have $F = (A \cup P^c)^o \cap P \subseteq (A \cup P^c) \cap P = (A \cap P) \cup (P^c \cap P) = (A \cap P) \cup \emptyset = A \cap P = A$, since $A \subseteq P$. Since F is open in P and $F \subseteq A$ it follows, from the definition of $\mathrm{Int}_P(A)$, that $F \subseteq \mathrm{Int}_P(A)$.

Let E be any open set in P with $E \subseteq A$. Choose an open set U in $\mathbf{R}^n$ such that $U \cap P = E$. Then we have $U = U \cap \mathbf{R}^n = U \cap (P \cup P^c) = (U \cap P) \cup (U \cap P^c) = E \cup (U \cap P^c) \subseteq A \cup P^c$, since $E \subseteq A$ and $U \cap P^c \subseteq P^c$. Since U is open in $\mathbf{R}^n$ and $U \subseteq A \cup P^c$ it follows that $U \subseteq (A \cup P^c)^o$. Since $E = U \cap P \subseteq U \subseteq (A \cup P^c)^o$ and $E \subseteq A \subseteq P$ we have $E \subseteq (A \cup P^c)^o \cap P = F$. Since $E \subseteq F$ for every open set E in P with $E \subseteq A$ it follows, from the definition of $\mathrm{Int}_P(A)$, that $\mathrm{Int}_P(A) \subseteq F$.