MATH 247 Calculus 3, Exercises for Chapter 3

1: (a) Show, from the definition of compactness, that the set $A = \mathbf{Q} \cap [0,1]$ is not compact.

(b) Show, from the definition of compactness, that the set $B = \left\{ \frac{n|n|}{1+n^2} \,\middle|\, n \in \mathbf{Z} \right\} \cup \{1, -1\}$ is compact.

(c) Show that the set $O_n(\mathbf{R}) = \left\{ A \in M_n(\mathbf{R}) \middle| A^T A = I \right\}$ is compact. Here, we are identifying $M_n(\mathbf{R})$ with $\mathbf{R}^{n^2}$, so that the dot product of two matrices is given by $A \bullet B = \sum_{k,\ell} A_{k,\ell} B_{k,\ell} = \text{trace}(B^T A)$.

2: For each of the following functions $f : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}$, find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that the limit does not exist.

(a) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ (b) $f(x,y) = \dfrac{x^2 y^3}{x^4 + y^6}$ (c) $f(x,y) = \dfrac{x^4 y^5}{x^8 + y^6}$

3: Let $f : A \subseteq \mathbf{R}^n \to B \subseteq \mathbf{R}^m$.

(a) Show that f is continuous if and only if $f^{-1}(F)$ is closed in A for every closed set F in B.

(b) Let E and F be closed sets in A with $E \cup F = A$. Let g be the restriction of f to E, and let h be the restriction of f to F. Show that f is continuous if and only if both g and h are continuous.

(c) Show that f is continuous if and only if for every $E \subseteq A$ we have $f\left(\overline{E}\right) \subseteq \overline{f(E)}$.

4: (a) Let $f : A \subseteq \mathbf{R}^n \to \mathbf{R}^m$. Show that if A is compact and f is continuous then f is uniformly continuous.

(b) Let $f : A \subseteq \mathbf{R}^n \to B \subseteq \mathbf{R}^m$. Show that if A is compact and f is continuous and bijective then f^{-1} is continuous.

(c) Let $\emptyset \neq A, B \subseteq \mathbf{R}^n$. Define the **distance** between A and B to be

$$d(A,B) = \inf \left\{ |x - y| \,\middle|\, x \in A, y \in B \right\}.$$

Show that if A is compact and B is closed and $A \cap B = \emptyset$ then $d(A,B) > 0$.

5: Let $A \subseteq \mathbf{R}^n$.

(a) For $a, b \in A$, write $a \sim b$ when there exists a continuous path in A from a to b. Show that $\sim$ is an equivalence relation on A (this means that for all $a, b, c \in A$ we have $a \sim a$, and if $a \sim b$ then $b \sim a$, and if $a \sim b$ and $b \sim c$ then $a \sim c$).

(b) Suppose that A is open and connected. Show that A is path connected.