MATH 247 Calculus 3, Solutions to the Exercises for Chapter 3

1: (a) Show, from the definition of compactness, that the set $A = \mathbf{Q} \cap [0,1]$ is not compact.

Solution: Let $a \in [0,1]$ with $a \notin \mathbf{Q}$ and note that a is a limit point of A because $\mathbf{Q}$ is dense in $\mathbf{R}$. For each $n \in \mathbf{Z}^+$ let $U_n = \overline{B}\big(a, \frac{1}{n}\big)^c = \big(-\infty, a - \frac{1}{n}\big) \cup \big(a + \frac{1}{n}, \infty\big)$, and let $S = \big\{U_n \big| n \in \mathbf{Z}^+\big\}$. Note that each U_n is open and we have $\bigcup_{n=1}^{\infty} U_n = \mathbf{R} \setminus \{a\}$, so S is an open cover of A. Let T be any nonempty finite subset of A, say $T = \big\{U_{n_1}, U_{n_2}, \cdots, U_{n_\ell}\big\}$ with $n_1 < n_2 < \cdots < n_\ell$. Note that $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ and so we have $\bigcup T = \bigcup_{k=1}^{\ell} U_{n_k} = U_{n_\ell} = \overline{B}\big(a, \frac{1}{n_\ell}\big)^c$. Since a is a limit point of A we have $B\big(a, \frac{1}{n}\big) \cap A \neq \emptyset$, hence $\overline{B}\big(a, \frac{1}{n}\big) \cap A \neq \emptyset$, and so A is not a subset of $\bigcup T$. Since no finite subset of S covers A, it follows that A is not compact.

(b) Show, from the definition of compactness, that the set $B = \left\{ \frac{n|n|}{1+n^2} \,\middle|\, n \in \mathbf{Z} \right\} \cup \{1, -1\}$ is compact.

Solution: Note that $\lim\limits_{n\to\infty} \dfrac{n|n|}{1+n^2} = 1$ and $\lim\limits_{n\to-\infty} \dfrac{n|n|}{1+n^2} = -1$. Let S be any open cover of B. Since S covers B and $\pm 1 \in B$ we can choose $V, W \in S$ such that $1 \in V$ and $-1 \in W$. Since V and W are open we can choose $r > 0$ such that $B(1, r) \subseteq V$ and $B(-1, r) \subseteq W$. Since $\lim\limits_{n\to\infty} \dfrac{n|n|}{1+n^2} = 1$ and $\lim\limits_{n\to\infty} \dfrac{n|n|}{1+n^2} = -1$ we can choose $N \in \mathbf{Z}^+$ such that for all $n \in \mathbf{Z}$, if $n \geq N$ then $\left|\frac{n|n|}{1+n^2} - 1\right| < r$ so that $\frac{n|n|}{1+n^2} \in V$ and if $n \leq -N$ then $\left|\frac{n|n|}{1+n^2} + 1\right| < r$ so that $\frac{n|n|}{1+n^2} \in W$. For each $n \in \mathbf{Z}$ with $-N < n < N$, choose $U_n \in S$ so that $\frac{n|n|}{1+n^2} \in U_n$. Then the set $T = \big\{U_n \big| -N < n < n\big\} \cup \{V, W\}$ is a finite subcover of S. Thus B is compact.

(c) Show that the set $O_n(\mathbf{R}) = \big\{A \in M_n(\mathbf{R}) \big| A^T A = I\big\}$ is compact. Here, we are identifying $M_n(\mathbf{R})$ with $\mathbf{R}^{n^2}$, so that the dot product of two matrices is given by $A \cdot B = \sum\limits_{k,\ell} A_{k,\ell} B_{k,\ell} = \operatorname{trace}(B^T A)$.

Solution: Note that for $A \in M_n(\mathbf{R})$ we have

$$A \in O_n(\mathbf{R}) \iff A^T A = I \iff (A^T A)_{k,l} = I_{k,l} \text{ for all } k,l \iff \sum_{i=1}^{n} A_{i,k} A_{i,l} = \delta_{k,l} \text{ for all } k,l,$$

where

$$\delta_{k,l} = \begin{cases} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l. \end{cases}$$

For each pair k,l, define $f_{k,l} : M_n(\mathbf{R}) \to \mathbf{R}$ by $f_{k,l}(A) = \sum\limits_{i=1}^{n} A_{i,k} A_{i,l} - \delta_{k,l}$. Note that each function $f_{k,l}$ is continuous since it is an elementary function on the n^2 variables $A_{i,j}$. We have

$$O_n(\mathbf{R}) = \big\{A \in M_r(\mathbf{R}) \big| f_{k,l}(A) = 0 \text{ for all } k,l\big\} = \bigcap_{k,l} \big\{A \in M_n(\mathbf{R}) \big| f_{k,l}(A) = 0\big\} = \bigcap_{k,l} f_{k,l}^{-1}(0).$$

Note that $f_{k,l}^{-1}(0)$ is the complement in $M_n(\mathbf{R})$ of the set $f_{k,l}^{-1}(\mathbf{R} \setminus \{0\})$. Since $\mathbf{R} \setminus \{0\}$ is open in $\mathbf{R}$ and each function $f_{k,l}$ is continuous, it follows that each set $f_{k,l}^{-1}(\mathbf{R} \setminus \{0\})$ is open, and hence each set $f_{k,l}^{-1}(0)$ is closed. Thus $O_n(\mathbf{R})$ is closed because it is the intersection of finitely many closed sets.

We claim that $O_n(\mathbf{R})$ is bounded. Let $A \in O_n(\mathbf{R})$. Let $u_1, u_2, \cdots, u_n$ be the columns of A. Note that

$$A^T A = \begin{pmatrix} {u_1}^T \\ \vdots \\ {u_n}^T \end{pmatrix} \big(u_1, \cdots, u_n\big) = \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_n \\ \vdots & & & \vdots \\ u_n \cdot u_1 & u_n \cdot u_2 & \cdots & u_n \cdot u_n \end{pmatrix}$$

and so

$$\begin{aligned} A^T A = I &\Longrightarrow (A^T A)_{k,k} = 1 \text{ for all } k \Longrightarrow u_k \cdot u_k = 1 \text{ for all } k \Longrightarrow |u_k| = 1 \text{ for all } k,l \\ &\Longrightarrow |A|^2 = \sum_{k=1}^{n} \sum_{i=1}^{n} (A_{i,k})^2 = \sum_{k=1}^{n} |u_k|^2 = \sum_{k=1}^{n} 1 = n. \end{aligned}$$

Thus for every $A \in O_n(\mathbf{R}^n)$ we have $|A| = \sqrt{n}$ and so $O_n(\mathbf{R})$ is bounded, as claimed. We have shown that $O_n(\mathbf{R})$ is closed and bounded, and so it is compact, by the Heine Borel Theorm (which we can apply because we are identifying $M_n(\mathbf{R})$ with $\mathbf{R}^{n^2}$).

2: For each of the following functions $f : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}$, find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ or show that the limit does not exist.

(a) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

Solution: Let $\theta \in \mathbf{R}$ and define $\alpha : \mathbf{R} \to \mathbf{R}^2$ by $\alpha(t) = (t\cos\theta, t\sin\theta)$. Then we have $\lim\limits_{t\to 0} \alpha(t) = (0,0)$ and $f(\alpha(t)) = \frac{t^2\cos^2\theta - t^2\sin^2\theta}{t^2\cos^2\theta + t^2\sin^2\theta} = \cos 2\theta$ for all $t \neq 0$, and so (by the Limits of Composites Theorem) if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed then it would be equal to $\cos 2\theta$. Since different choices of θ yield different values for the limit, the limit cannot exist.

(b) $f(x,y) = \dfrac{x^2 y^3}{x^4 + y^6}$

Solution: Consider the graph $z = f(x,y)$. The level set $y = c > 0$ is given by $z = g(x) = f(x,c) = \frac{c^3 x^2}{x^4 + c^6}$. Then

$$z' = g'(x) = \tfrac{c^3(2x(x^4+c^6) - (x^2)(4x^3))}{(x^4+c^6)^2} = \tfrac{c^3(2x)(c^6 - x^4)}{(x^4+c^6)^2}\,,$$

so we have $z' = 0$ when $x = 0$ and when $x = \pm c^{3/2}$. When $x = 0$ we have $z = 0$ and when $x = \pm c^{3/2}$ we have $z = \frac{c^3 \cdot c^3}{c^6 + c^6} = \frac{1}{2}$. The graph $z = f(x,y)$ with $y > 0$ has a maximum ridge of height $z = \frac{1}{2}$ along $x = \pm y^{3/2}$, that is $x^2 = y^3$.

Define $\alpha : \mathbf{R} \to \mathbf{R}^2$ by $\alpha(t) = (0,t)$. Then $\lim\limits_{t\to 0} \alpha(t) = (0,0)$ and $f(\alpha(t)) = 0$ for all $t \neq 0$, and so (by The Limits of Composites Theorem) if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed then it would be equal to 0. Define $\beta : \mathbf{R} \to \mathbf{R}^2$ by $\beta(t) = (t^3, t^2)$. Then $\lim\limits_{t\to 0} \beta(t) = (0,0)$ and $f(\beta(t)) = \frac{t^6 \cdot t^6}{t^{12} + t^{12}} = \frac{1}{2}$ for all $t \neq 0$, and so if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed then it would be equal to $\frac{1}{2}$. Thus the limit cannot exist.

(c) $f(x,y) = \dfrac{x^4 y^5}{x^8 + y^6}$

Solution: Recall that for all $u, v \in \mathbf{R}$ we have $0 \leq (|u| - |v|)^2 = u^2 - 2|uv| + v^2$ and so $|uv| \leq \frac{1}{2}(u^2 + v^2)$. It follows that for all $(x,y) \neq (0,0)$ we have

$$\big|f(x,y) - 0\big| = \big|\tfrac{x^4 y^5}{x^8 + y^6}\big| = \tfrac{|x^4 y^3| y^2}{x^8 + y^6} \leq \tfrac{\frac{1}{2}(x^8 + y^6) y^2}{x^8 + y^6} = \tfrac{1}{2}\, y^2.$$

Given $\epsilon > 0$ choose $\delta = \sqrt{2\epsilon}$. Then for all x, y with $0 < \big|(x,y)\big| < \delta$ we have $0 < x^2 + y^2 < \delta^2$ and so

$$\big|f(x,y) - 0\big| \leq \tfrac{1}{2}\, y^2 \leq \tfrac{1}{2}(x^2 + y^2) < \tfrac{1}{2}\delta^2 = \epsilon.$$

3: Let $f : A \subseteq \mathbf{R}^n \to B \subseteq \mathbf{R}^m$.

(a) Show that f is continuous if and only if $f^{-1}(F)$ is closed in A for every closed set F in B.

Solution: We already know that f is continuous if and only if $f^{-1}(E)$ is open in A for every open set E in B. Suppose that f is continuous. Let F be a closed set in B. Then $B \setminus F$ is open in B and so $f^{-1}(B \setminus F)$ is open in A and hence $A \setminus f^{-1}(B \setminus F)$ is closed in A. But notice that $f^{-1}(F) = A \setminus f^{-1}(B \setminus F)$ because for $a \in A$ we have

$$a \in f^{-1}(F) \iff f(a) \in F \iff f(a) \notin B \setminus F \iff a \notin f^{-1}(B \setminus F) \iff a \in A \setminus f^{-1}(B \setminus F).$$

Thus $f^{-1}(F)$ is closed in A for every closed set F in B.

Conversely, suppose that $f^{-1}(F)$ is closed in A for every closed set F in B. Let E be an open set in B. Then $B \setminus E$ is closed in B, hence $f^{-1}(B \setminus E)$ is closed in B, and so $A \setminus f^{-1}(B \setminus E)$ is open in A. But notice that $f^{-1}(E) = A \setminus f^{-1}(B \setminus E)$, as above. This shows that that $f^{-1}(E)$ is open in A for every open set E in B, and so f is continuous.

(b) Let E and F be closed sets in A with $E \cup F = A$. Let g be the restriction of f to E, and let h be the restriction of f to F. Show that f is continuous if and only if both g and h are continuous.

Solution: We begin by remarking that when $S \subseteq A \subseteq \mathbf{R}^n$, the open sets in S are the sets of the form $L \cap S$ with L being an open set in A. Indeed when L is open in A we can choose an open set U in $\mathbf{R}^n$ such that $L = U \cap A$, and then we have $L \cap S = (U \cap A) \cap S = U \cap S$ since $S \subseteq A$. On the other hand, when E is open in S we can choose an open set U in $\mathbf{R}^n$ such that $E = U \cap S$ and then the set $L = U \cap A$ is open in A with $L \cap S = (U \cap A) \cap S = U \cap S = E$. Similarly, the closed sets in S are the sets of the form $K \cap S$ with K being a closed set in A.

Suppose $f : A \to B$ is continuous. We claim that the restriction of f to *any* subset $S \subseteq A$ is continuous. Let $S \subseteq A$ and let $p : S \subseteq A \to B$ be the restriction of f to S. Let E be an open set in B. Then $f^{-1}(E)$ is open in A and so $S \cap f^{-1}(E)$ is open in S. But notice that $p^{-1}(E) = S \cap f^{-1}(E)$ since for $a \in A$ we have

$$\begin{aligned} a \in p^{-1}(E) &\iff a \in S \text{ and } p(a) \in E \iff a \in S \text{ and } f(a) \in E \\ &\iff a \in S \text{ and } a \in f^{-1}(E) \iff a \in S \cap f^{-1}(E). \end{aligned}$$

This shows that $p^{-1}(E)$ is open in S for every open set E in B, and so p is continuous in S.

Conversely, suppose that both of the two restrictions g and h are continuous. Let C be a closed set in B. Then $g^{-1}(C)$ is closed in E and $h^{-1}(C)$ is closed in F. Since $g^{-1}(C)$ is closed in E we can choose a closed set K in A so that $g^{-1}(C) = E \cap K$. Since E and K are both closed in A, it follows that $g^{-1}(C)$ is closed in A. Similarly, since $h^{-1}(C)$ is closed in F and F is closed in A, it follows that $h^{-1}(C)$ is closed in A. Since $g^{-1}(C)$ and $h^{-1}(C)$ are both closed in A, their union $g^{-1}(C) \cup h^{-1}(C)$ is closed in A. But notice that $f^{-1}(C) = g^{-1}(C) \cup h^{-1}(C)$ because for $a \in A$ we have

$$\begin{aligned} a \in f^{-1}(C) &\iff a \in A \text{ and } f(a) \in C \iff a \in E \cup F \text{ and } f(a) \in C \\ &\iff \big(a \in E \text{ and } f(a) \in C\big) \text{ or } \big(a \in F \text{ and } f(a) \in C\big) \\ &\iff \big(a \in E \text{ and } g(a) \in C\big) \text{ or } \big(a \in F \text{ and } h(a) \in C\big) \\ &\iff a \in g^{-1}(C) \text{ or } a \in h^{-1}(C). \end{aligned}$$

(c) Show that f is continuous if and only if for every $E \subseteq A$ we have $f\big(\overline{E}\big) \subseteq \overline{f(E)}$.

Solution: Suppose that f is continuous. Let $E \subseteq A$. Let $b \in f\big(\overline{E}\big)$, say $b = f(a)$ where $a \in A \cap \overline{E}$. We must show that $b \in \overline{f(E)}$. Let $r > 0$. Since $B_B(b, r)$ is open in B and f is continuous, $f^{-1}\big(B_B(b, r)\big)$ is open in A, so we can choose $s > 0$ so that $B_A(a, s) \subseteq f^{-1}\big(B_B(b, r)\big)$. Since $a \in A \cap \overline{E}$, we have $B_A(a, s) \cap E \neq \emptyset$, so we can choose a point $c \in B_A(a, s) \cap E$. Since $c \in B_A(a, s) \subseteq f^{-1}\big(B_B(b, r)\big)$ we have $f(c) \in B_B(b, r)$, and since $c \in E$ we have $f(c) \in f(E)$, and so $f(c) \in B_B(b, r) \cap f(E)$. Thus $B_B(b, r) \cap f(E) \neq \emptyset$ for all $r > 0$, so $b \in \overline{f(E)}$, as required.

Conversely, suppose that for every $E \subseteq A$ we have $f\big(\overline{E}\big) \subseteq \overline{f(E)}$. Let $K \subseteq B$ be closed in B. We claim that $f^{-1}(K)$ is closed in A. Let $C = f^{-1}(K)$. Note that $f(C) \subseteq K$. Let $x \in \overline{C}$. Then $f(x) \in f(\overline{C}) \subseteq \overline{f(C)} \subseteq \overline{K} = K$ and so $x \in f^{-1}(K) = C$. Thus $\overline{C} \subseteq C$. Of course we also have $C \subseteq \overline{C}$, so $C = \overline{C}$, and so C is closed, as claimed. Thus f is continuous.

4: (a) Let $f : A \subseteq \mathbf{R}^n \to \mathbf{R}^m$. Show that if A is compact and f is continuous then f is uniformly continuous.

Solution: Suppose that A is compact and f is continuous. Let $\epsilon > 0$. For each $a \in A$, since f is continuous at a we can choose $\delta_a > 0$ such that $|x-a| < 2\delta_a \Longrightarrow \big|f(x)-f(a)\big| < \frac{\epsilon}{2}$. Let $S = \big\{B(a,\delta_a)\big|a \in A\big\}$ and note that S is an open cover of A. Since A is compact, we can choose a finite subcover T of S, say $T = \big\{B(a_k,\delta_{a_k})\big|1 \le k \le \ell\big\}$. Let $\delta = \min\big\{\delta_{a_k}\big|1 \le k \le \ell\big\}$. Let $x, y \in A$ with $|x-y| < \delta$. Since T covers A we can choose an index k such that $x \in B(a_k,\delta_{a_k})$. Since $|x-a_k| < \delta_{a_k}$ and $|x-y| < \delta \le \delta_{a_k}$ we have $|y-a_k| \le 2\delta_{a_k}$. Since $|x-a_k| < 2\delta_{a_k}$ and $|y-a_k| < 2\delta_{a_k}$ we have $\big|f(x)-f(a_k)\big| < \frac{\epsilon}{2}$ and $\big|f(y)-f(a_k)\big| < \frac{\epsilon}{2}$ and hence $\big|f(x)-f(y)\big| < \epsilon$.

(b) Let $f : A \subseteq \mathbf{R}^n \to B \subseteq \mathbf{R}^m$. Show that if A is compact and f is continuous and bijective then f^{-1} is continuous.

Solution: Suppose that A is compact and f is continuous and bijective, and let $g = f^{-1} : B \to A$. Let E be a closed set in A. By the Heine-Borel Theorem, A is closed and bounded. Since E is closed in A we can choose a closed set K in $\mathbf{R}^n$ such that $E = K \cap A$ (by Theorem 2.31). Since K and A are closed in $\mathbf{R}^n$, so is $E = K \cap A$ (by Theorem 2.14). Since $E \subseteq A \subseteq \mathbf{R}^n$ with E closed and A compact, it follows that E is compact (by Theorem 2.28). Since E is compact and f is continuous, it follows that $f(E)$ is compact (by Theorem 3.37 Part 2) hence $f(E)$ closed (by the Heine-Borel Theorem). Since f and g are inverses, we have $g^{-1}(E) = f(E)$, which is closed. Since $g^{-1}(E)$ is closed for every closed set E in A, it follows that g is continuous (by Theorem 3.36 Part 2, proved in Problem 3 a).

(c) Let $\emptyset \neq A, B \subseteq \mathbf{R}^n$. Define the **distance** between A and B to be

$$d(A,B) = \inf\big\{|x-y|\,\big|\,x \in A, y \in B\big\}.$$

Show that if A is compact and B is closed and $A \cap B = \emptyset$ then $d(A,B) > 0$.

Solution: Since B is closed, hence $B^c = \mathbf{R}^n \setminus B$ is open, for each $a \in A$ we can choose $r_a > 0$ so that $B(a, 2r_a) \subseteq B^c$. The set $S = \big\{B(a,r_a)\big|a \in A\big\}$ is an open cover of A. Since A is compact, we can choose a finite subcover $T \subseteq S$, say $T = \big\{B(a_1,r_{a_1}), B(a_2,r_{a_2}), \cdots, B(a_\ell,r_{a_\ell})\big\}$ where each $a_k \in A$. Let $r = \min\{r_{a_1}, r_{a_2}, \cdots, r_{a_\ell}\}$. We claim that $d(A,B) \ge r$. Let $x \in A$ and $y \in B$. Since T covers A, we can choose an index k so that $x \in B(a_k, r_{a_k})$ hence $|x-a_k| < r_{a_k}$. Since $y \in B$ and $B(a_k, 2r_{a_k}) \subseteq B^c$ we must have $|y-a_k| \ge 2r_{a_k}$. By the Triangle Inequality, $|y-a_k| \le |y-x| + |x-a_k|$ hence $|y-x| \ge |y-a_k| - |x-a_k| \ge 2r_{a_k} - r_{a_k} = r_{a_k} \ge r$. Since $|y-x| \ge r$ for all $x \in A$ and $y \in B$ we have $d(A,B) = \inf\big\{|y-x|\,\big|\,x \in A, y \in B\big\} \ge r$, as claimed.

5: Let $A \subseteq \mathbf{R}^n$.

(a) For $a, b \in A$, write $a \sim b$ when there exists a continuous path in A from a to b. Show that $\sim$ is an equivalence relation on A (this means that for all $a, b, c \in A$ we have $a \sim a$, and if $a \sim b$ then $b \sim a$, and if $a \sim b$ and $b \sim c$ then $a \sim c$).

Solution: Let $a, b, c \in A$. We have $a \sim a$ because we can define $\alpha : [0,1] \to A$ by $\alpha(t) = a$ for all t, and then α is continuous with $\alpha(0) = a$ and $\alpha(1) = a$, so α is a path in A from a to a.

Suppose that $a \sim b$. Let α be a path in A from a to b, so $\alpha : [0,1] \to A$ is continuous with $\alpha(0) = a$ and $\alpha(1) = b$. Define $\beta : [0,1] \to A$ by $\beta(t) = \alpha(1-t)$. Note that β is continuous since it is the composite of the continuous map α with the continuous map $s : [0,1] \to [0,1]$ given by $s(t) = 1 - t$, and note that we have $\beta(0) = \alpha(1) = b$ and $\beta(1) = \alpha(0) = a$. Thus β is a path in A from b to a and so $b \sim a$.

Finally, suppose that $a \sim b$ and $b \sim c$. Let α be a path from a to b in A and let β be a path from b to c in A. Define $\gamma : [0,1] \to A$ by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{, for } 0 \le t \le \frac{1}{2}, \\ \beta(2t-1) & \text{, for } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that $\gamma(0) = \alpha(0) = a$, $\gamma\left(\frac{1}{2}\right) = \alpha(1) = \beta(0) = b$, and $\gamma(1) = \beta(1) = c$. Gamma is continuous by Problem 3(b), because the sets $E = \left[0, \frac{1}{2}\right]$ and $F = \left[\frac{1}{2}, 1\right]$ are closed in $[0,1]$ with $E \cup F = [0,1]$, and the restriction of γ to E is given by $\alpha(2t)$, which is continuous (being the composite of two continuous functions), and the restriction of γ to F is given by $\beta(2t-1)$, which is also continuous.

(b) Suppose that A is open and connected. Show that A is path connected.

Solution: The empty set is open, connected and path-connected (vacuously). Suppose $A \neq \emptyset$ and let $a \in A$. Let

$$E = \left\{ b \in A \middle| a \sim b \right\}.$$

We claim that E is open in A. Let $b \in E$. Since $b \in A$ and A is open in $\mathbf{R}^n$, we can choose $r > 0$ so that $B(b,r) \subseteq A$. Let $c \in B(b,r)$. Since $b \in E$ we have $a \sim b$. Since $c \in B(b,r) \subseteq A$ we have $b \sim c$, indeed we can define $\alpha : [0,1] \to B(b,r) \subseteq A$ by $\alpha(t) = b + t(c-b)$ and then α is continuous (since it elementary), and $\alpha(0) = b$ and $\alpha(1) = c$, and $\alpha(t) \in B(b,r)$ for all $t \in [0,1]$ because $\big|\alpha(t) - b\big| = \big|t(c-b)\big| = |t||c-b| \le |c-b| < r$. Since $a \sim b$ and $b \sim c$ we have $a \sim c$ by Part (a). Since $a \sim c$ we have $c \in E$, hence $B(b,r) \subseteq E$. This shows that E is open in $\mathbf{R}^n$ hence also in A.

We claim that E is also closed in A. Let $b \in A \setminus E$. Since $b \in A$ and A is open in $\mathbf{R}^n$, we can choose $r > 0$ so that $B(b,r) \subseteq A$. Let $c \in B(b,r)$. Since $b \notin E$ we have $a \not\sim b$. Since $c \in B(b,r) \subseteq A$ we have $b \sim c$, as above. It follows from Part (a) that $a \not\sim c$ since otherwise we would have $a \sim c$ and $c \sim b$ and hence $a \sim b$. Since $c \not\sim a$ we have $c \in A \setminus E$. Thus $B(b,r) \subseteq A \setminus E$. This shows that $A \setminus E$ is open (both in $\mathbf{R}^n$ and in A) so that E is closed in A.

Since A is connected, the only subsets of A which are both open and closed are $\emptyset$ and A. Since E is both open and closed we must have $E = \emptyset$ or $E = A$. Since $a \sim a$ we have $a \in E$ so $E \neq \emptyset$ and so $E = A$. Since $A = E = \left\{ b \in A \middle| a \sim b \right\}$ we have $a \sim b$ for every $b \in A$. Thus A is path connected.