MATH 247 Calculus 3, Solutions to the Exercises for Chapter 4

1: (a) Find an implicit and an explicit equation for the tangent line to the parametric curve $(x,y)=(\cos t,\sin 2t)$ at the point where $t=\frac{\pi}{3}$.

Solution: Let $f(t)=(\cos t,\sin 2t)$ and note that $f'(t)=(-\sin t,2\cos 2t)$. The required tangent line is the line through the point $f\left(\frac{\pi}{3}\right)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ in the direction of the vector $f'\left(\frac{\pi}{3}\right)=\left(-\frac{\sqrt{3}}{2},-1\right)$, so the line is given parametrically by $(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)+t\left(\frac{\sqrt{3}}{2},1\right)$. A normal vector is given by $\left(1,-\frac{\sqrt{3}}{2}\right)$, so the equation can be written as $x-\frac{\sqrt{3}}{2}y=c$. Put in the point $(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ to get $c=\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}=-\frac{1}{4}$. Thus the line has equation $x-\frac{\sqrt{3}}{2}y=-\frac{1}{4}$, so it is given explicitly by the equation $x=-\frac{1}{4}+\frac{\sqrt{3}}{2}y$ or by the equation $y=\frac{2}{\sqrt{3}}x+\frac{1}{2\sqrt{3}}$.

(b) The position of a fly at time t is given by $(x,y,z)=(t,t^2,1+t^3)$. A light shines down on the fly from the point $(0,0,3)$ and casts a shadow on the xy-plane. Find the position and the velocity of the shadow of the fly at time $t=1$.

Solution: When the fly is at the point (x,y,z) with $z<3$, let us find a formula for the position $(u,v,0)$ of the shadow. The line from the light at $(0,0,3)$ to the fly at (x,y,z) has parametric equation

$$(u,v,w)=(0,0,3)+s\left((x,y,z)-(0,0,3)\right)=\left(sx\,,\,sy\,,\,3+s(z-3)\right).$$

The shadow is at the point where this line touches the xy-plane, that is the point where $w=0$. To get $w=0$, we need $3+s(z-3)=0$, and so $s=\dfrac{3}{3-z}$, and then $u=sx=\dfrac{3x}{3-z}$ and $v=st=\dfrac{3y}{3-z}$. This shows that when the fly is at the point $(x,y,z)=(t,t^2,1+t^3)$, the shadow is at the point

$$(u,v)=(u(t),v(t))=\left(\frac{3x}{3-z},\frac{3y}{3-z}\right)=\left(\frac{3t}{2-t^3},\frac{3t^2}{2-t^3}\right)$$

and its velocity is

$$(u'(t),v'(t))=\left(\frac{(3)(2-t^3)-(3t)(-3t^2)}{(2-t^3)^2},\frac{(6t)(2-t^3)-(3t^2)(-3t^2)}{(2-t^3)^2}\right)=\left(\frac{6+6t^3}{(2-t^3)^2},\frac{12t+3t^4}{(2-t^3)^2}\right).$$

In particular, we have $\left(u(1),v(1)\right)=(3,3)$ and $\left(u'(1),v'(1)\right)=(12,15)$.

2: Let S be the parametric surface $(x,y,z)=f(s,t)=\left(\frac{s}{t},\sqrt{s+t},st\right)$.

(a) Find the derivative matrix $Df(s,t)$.

Solution: The derivative matrix is

$$Df(s,t)=\begin{pmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t}\end{pmatrix}=\begin{pmatrix}\frac{1}{t} & -\frac{s}{t^2}\\ \frac{1}{2\sqrt{s+t}} & \frac{1}{2\sqrt{s+t}}\\ t & s\end{pmatrix}.$$

(b) Find a parametric equation for the tangent plane to S at the point where $(s,t)=(2,2)$.

Solution: The tangent plane is given parametrically by

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=L(s,t)=f(2,2)+Df(2,2)\begin{pmatrix}s-2\\ t-2\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 4\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{4} & \frac{1}{4}\\ 2 & 2\end{pmatrix}\begin{pmatrix}s-2\\ t-2\end{pmatrix}$$

that is by

$$(x,y,z)=(1,2,4)+\left(\tfrac{1}{2},\tfrac{1}{4},2\right)(s-2)+\left(-\tfrac{1}{2},\tfrac{1}{4},2\right)(t-2).$$

Alternatively, by introducing new parameters u and v with $s-2=4u$ and $t-2=4v$, we have

$$(x,y,z)=(1,2,4)+(2,1,8)\,u+(-2,1,8)\,v.$$

(c) Find an implicit equation for the tangent plane to S at the point where $(s,t)=(2,2)$.

Solution: The plane has normal vector $(2,1,8)\times(-2,1,8)=(0,-32,4)$. We can multiply this vector by $-\frac{1}{4}$ to get the simpler normal vector $(0,8,-1)$, so the equation of the plane is of the form $0x+8y-1z=c$ for some constant c. Put in the point $(x,y,z)=(1,2,4)$ to get $c=12$. Thus the tangent plane is given implicitly by $8y-z=12$ (or explicitly $z=8y-12$).

3: Let C be the curve of intersection of the two surfaces $z = x^2 - 2y$ and $z = 2x^2 + y^2$. Find a parametric equation for the tangent line L to the curve C at the point $(-1, -1, 3)$ using each of the following two methods.

(a) Find the equation of the tangent plane to each of the two surfaces at $(-1, -1, 3)$, then solve the two equations to obtain a parametric equation for L.

Solution: Note that the first surface is given explicitly by $z = f(x, y) = x^2 - 2y$. We have $\frac{\partial f}{\partial x}(x, y) = 2x$ and $\frac{\partial f}{\partial y}(x, y) = -2$. The equation of the tangent plane is

$$z = f(-1,-1) + \tfrac{\partial f}{\partial x}(-1,-1)(x+1) + \tfrac{\partial f}{\partial y}(-1,-1)(y+1) = 3 - 2(x+1) - 2(y+1) = -2x - 2y - 1.$$

The second surface is given explicitly by $z = g(x, y) = 2x^2 + y^2$. We have $\frac{\partial g}{\partial x} = 4x$ and $\frac{\partial g}{\partial y} = 2y$ so the equation of the tangent plane is

$$z = g(-1,-1) + \tfrac{\partial g}{\partial x}(-1,-1)(x+1) + \tfrac{\partial g}{\partial y}(-1,-1)(y+1) = 3 - 4(x+1) - 2(y+1) = -4x - 2y - 3.$$

The equations of the two planes can be written as $2x + 2y + z = -1$ and $4x + 2y + z = -3$. We solve these two equations using Gauss-Jordan elimination. We have

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 4 & 2 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

so the solution is

$$(x, y, z) = \big(-1, \tfrac{1}{2}, 0\big) + \big(0, -\tfrac{1}{2}, 1\big)\, t\,.$$

(b) Find a parametric equation for C, then use this parametric equation to find a parametric equation for the tangent line L.

Solution: For any point (x, y, z) which lies in the intersection, we must have $z = x^2 - 2y$ and $z = 2x^2 + y^2$, and so $x^2 - 2y = 2x^2 + y^2$, that is $x^2 + y^2 + 2y = 0$. Complete the square to rewrite this as $x^2 + (y+1)^2 = 1$, and we see that (x, y) lies on the circle centered at $(0, -1)$ of radius 1. This circle is given parametrically by $(x, y) = \big(\cos t, \sin t - 1\big)$. Put $x = \cos t$ and $y = \sin t - 1$ back into the equation $z = x^2 - 2y$ to get $z = \cos^2 t - 2\sin t + 2$. Thus the curve of intersection is given parametrically by

$$(x, y, z) = \big(\cos t\,,\ \sin t - 1\,,\ \cos^2 t - 2\sin t + 2\big)\,.$$

The tangent vector at each point is given by $(x', y', z') = \big(-\sin t, \cos t, -2\sin t\cos t - 2\cos t\big)$. Notice that when $t = \pi$ we have $(x, y, z) = (-1, -1, 3)$ and $(x', y', z') = (0, -1, 2)$, so the tangent line at the point $(x, y, z) = (-1, -1, 3)$ is given parametrically by

$$(x, y, z) = (-1, -1, 3) + (0, -1, 2)\, t\,.$$

4: (a) Let P be the tangent plane to the surface given by $z = 4x^2 - 8xy + 5y^2$ at the point where $(x, y) = (2, 1)$. Find the line of intersection of P with the xy-plane.

Solution: The surface is given explicitly by $z = f(x, y) = 4x^2 - 8xy + 5y^2$. We have $\frac{\partial f}{\partial x} = 8x - 8y$ and $\frac{\partial f}{\partial y} = -8x + 10y$, so the equation of the tangent plane P is

$$z = f(2,1) + \tfrac{\partial f}{\partial x}(2,1)(x-2) + \tfrac{\partial f}{\partial y}(2,1)(y-1) = 5 + 8(x-2) - 6(y-1) = 8x - 6y - 5.$$

To find the intersection of this plane with the xy-plane, put in $z = 0$ to get $8x - 6y = 5$.

(b) Find the equation of the tangent plane to the surface given by $e^{x+z} = \sqrt{x^2y + z}$ at the point $(1, 2, -1)$.

Solution: The surface is given implicitly by $g(x, y, z) = 0$ where $g(x, y, z) = e^{x+z} - \sqrt{x^2y + z}$. We have

$$\frac{\partial g}{\partial x} = e^{x+z} - \frac{xy}{\sqrt{x^2y+z}}\ ,\ \frac{\partial g}{\partial y} = -\frac{x^2}{2\sqrt{x^2y+z}} \text{ and } \frac{\partial g}{\partial z} = e^{x+z} - \frac{1}{2\sqrt{x^2y+z}}$$

so that

$$\tfrac{\partial g}{\partial x}(1,2,-1) = e^0 - \tfrac{2}{\sqrt{1}} = -1\ ,\ \tfrac{\partial g}{\partial y}(1,2,-1) = -\tfrac{1}{2\sqrt{1}} = -\tfrac{1}{2} \text{ and } \tfrac{\partial g}{\partial z}(1,2,-1) = e^0 - \tfrac{1}{2\sqrt{1}} = \tfrac{1}{2}.$$

Thus the equation of the tangent plane is

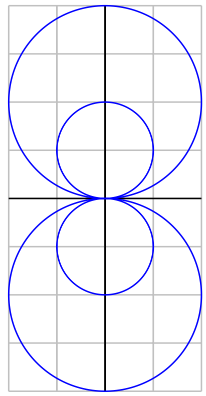
$$0 = \tfrac{\partial g}{\partial x}(1,2,-1)(x-1) + \tfrac{\partial g}{\partial y}(1,2,-1)(y-2) + \tfrac{\partial g}{\partial z}(1,2,-1)(z+1) = -(x-1) - \tfrac{1}{2}(y-2) + \tfrac{1}{2}(z+1).$$

Multiply both sides by -2 to get $0 = 2(x-1) + (y-2) + (z+1) = 2x + y - z - 5$. Thus the tangent plane is given implicitly by $2x + y - z = 5$ (or explicitly by $z = 2x + y - 5$).

5: Let S be the surface $2yz = x^2 + y^2$.

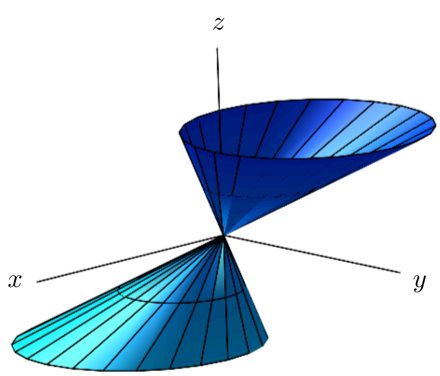
(a) Sketch the level sets $z = -2, -1, 0, 1, 2$ for the surface S (in other words, sketch the curve of intersection of S with the each of the planes $z = -2, -1, 0, 1, 2$).

Solution: The level set $z = -2$ is the curve $x^2 + y^2 = -4y$, that is $x^2 + y^2 + 4y = 0$ or, by completing the square, $x^2 + (y+2)^2 = 4$, so it is the circle centered at $(0, -2)$ of radius 2. In general, the level curve $z = c$ is the curve $x^2 + y^2 - 2cy = 0$ or $x^2 + (y-c)^2 = c^2$, which is the circle centered at $(0, c)$ of radius $|c|$. When $c = 0$, the level set consists only of the origin. The level sets are shown below.



(b) Sketch the surface S.

Solution: To sketch the surface, we draw each of the level sets $z = c$ at height c. It also helps to find the level sets $x = 0$ and $y = 0$. When $x = 0$ (that is in the yz-plane) we get the curve $2yz = y^2$, that is $y^2 - 2yz = 0$ or $y(y - 2z) = 0$, which is the union of the two lines $y = 0$ and $y = 2z$ in the yz-plane. When $y = 0$ (that is in the xz-plane) we get $x^2 = 0$, that is the line $x = 0$ in the xz-plane.



(c) Find the equation of the tangent plane to S at the point $(3, 1, 5)$.

Solution: Note that S is given implicitly by $g(x, y, z) = 0$ where $g(x, y, z) = x^2 + y^2 - 2yz$ and that we have $g(3, 1, 5) = 0$. We have $Dg = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = (2x, 2y - 2z, -2y)$ so that $Dg(3, 1, 5) = (6, -4, -2)$. The equation of the tangent plane is

$$0 = Dg(3,1,5)\begin{pmatrix} x-3 \\ y-1 \\ z-5 \end{pmatrix} = 6(x-3) - 4(y-1) - 2(z-5) = 6x - 4y - 2z - 4.$$

We can also write the equation explicitly as $z = 3x - 2y - 2$.